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On Market Economies: How Controllable Constructs Become Complex

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Since Lėon Walras neoclassical economists hold an inalterable belief in a unique and stable equilibrium for the economic system which however remains to this day unobservable. Yet that belief is the corner stone of other theories such as the 'Efficient Market Hypothesis' as well as the philosophy of neo-liberalism, whose outcomes are also shown to be flawed by recent events. A modern market economy is obviously an input/output nonlinear controllable construct. However, this paper examines four such models of increasing complexity, including the affine nonlinear feedback H_∞ control, to show that the 'data requirement' precludes all attempts at the empirical verification of the existence of a stable equilibrium. If equilibria of complex nonlinear deterministic systems are most likely unstable, multiple or deterministically chaotic depending on their parameter values and uncertainties, then society should impose limits on the state space and focus on endurable patterns thrown-off by such systems.

Keywords: *Equilibrium, nonlinearity, controllability, nonlinear feedback, H_∞-control, complexity.*

JEL Classification: *C61, C62, C68, C68, D57, D58.*

1. Introduction

The basic assumptions of neo-classical economics are well-known, but this paper is mainly concerned with a subset of these. For example, it is assumed that individuals and firms *optimize* under constraints; that agents are *rational* and always have *rational expectations*; that *the more connected are networks of individual participants the less risky, stable and robust is the economic system,* etc. Undoubtedly, the most misleading of these assumptions, i.e., the one which is our main focus here, is the *claim that market economies tend toward stable equilibria* (or an optimal Pareto state of balance). It is understood that such systems may be found away from their equilibrium points as a result of exogenous shocks, but they will inexorably return to their equilibrium on their own power. As a consequence, therefore, policy gurus of neo-liberalism propagate other beliefs to the effect that markets should be allowed to make all the major economic, social, and political decisions; that the state should refrain from any attempt to control markets, or that even public institutions should be handled over to corporations for the sake of greater efficiency, etc.

Strangely enough, after more than a century and a half not a single one of these assumptions has found empirical support. Beside the observational judgment to the effect that economic agents are generally self-interested and have strong monotone preference, most of those assumptions are wishful pronouncements. Yet,

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the equilibrium assumption, for example, though unobservable, is the foundations of both the 'Efficient market Hypothesis' and neo-liberalism that have had and continue to have very undesirable outcomes, in the Western economies in particular. These counterfactual results and the inability to sustain such a pivotal assumption do not bode well for that social science. It would, therefore, be useful to reexamine a few models of market economies so as to see why the stable equilibrium assumption has always escaped empirical verification regardless of assumed models' characteristics.

This paper consists of two parts. The first reexamines two linear models. That is the Walrasian pure exchange model, and a controllable linear time invariant model. The former is incomplete but is simple to analyze; its merit lies in the fact that it provides the first mathematical expression of the stable equilibrium assumption. The latter stumbles on observability and linearity. More realistically, it is proper to suppose at the outset that a modern market is a complex construct designed to facilitate exchange, which is more natural. It should therefore be controllable. Hence, Part II appeals to the recent but well characterized L_2- gain analysis of nonlinear systems and nonlinear feedback H-infinity control to examine two classes of nonlinear models, one in the non-affine category and the other in the affine categories. All four cases show that the equilibrium of market economies may well exist in theory, but will remain forever unobservable due to the *complexity* of markets and /or due to the formidable *data requirement* for such an endeavor. In the concluding remarks, we will then offer a few suggestions on how to navigate in complex systems.

2. Part I

In this section, we will review the Walrasian pure exchange (WPE) model and a controllable linear time invariant (LTI) model. We will show, on the one hand, that the equilibrium of the WPE model can easily be inferred but not easily demonstrated empirically. Though naïve and unrealistic, it provides nevertheless the justification for a set of beliefs that may still be blocking progress in the development of the neo-classical theory of economics. The LTI model, on the other hand, completely shunts the unavoidable complexity of real markets. Despite the unrealism of both models, however, their analyses are still worthwhile, for they clearly show that their data requirement precludes all attempts to characterize a stable equilibrium.

2.1. The Beginning

The model conceived by Walras from observing the functioning of the 'Bourse de Paris' is that of a pure exchange economy. It supposes there are i consumers ($i \varepsilon m$) of j goods ($j \varepsilon n$). Each consumer devotes a fraction α^i_j of his or her budget (B) to good j such that $\Sigma_j\alpha^i_j = 1$. The budget of i comes from the sale of endowments ω^i_j such that the demand of i for good j is $x^i_j = \alpha^i_j (B) /p_j$, where p_j is the price of j. Walras supposed a one period market. Hence in the neighborhood of the equilibrium point, we have a first-order linear differential equation:

$$\begin{aligned} \dot{x} &= dg\,(1/x_j)\,[A - dg\,(\Sigma_j\omega_j)]\,x \\ &= dg\,(1/x_j)\,[M]\,x, \end{aligned} \tag{1}$$

where $x \varepsilon X \varepsilon \Re^n$ is the state vector, and $[A - dg\,(\Sigma\omega_j)] = M_{n \times n}$. For the derivation of (1), (see Dominique, 2008).

Equation (1) is an input/output construct (with inputs $\Sigma_j\,\omega_j$ and output $x \varepsilon X \varepsilon \Re^n$) driven by incentives to minimize excess demand of all goods j.

For a solution, Walras posited a tâtonnement process controlled by an auctioneer. Had he taught of an exogenous supply rate for a sequence market, (1) would have been written as,

$$\dot{x} = M x,\ \ x \varepsilon X \varepsilon \Re^n_+ ,\ x(0) = x_o > 0 \text{ as initial condition,} \tag{2}$$

and (2) would have been represented by a linear system of differential equations whose solution is:

$$x(t) = e^{M t}\, x_{o}, \tag{3}$$

where $e^{M t}$ is an n x n matrix function defined by its Taylor series, provided of course that M is invertible.

If M were a real invertible matrix of order n x n, it would be called a Metzler matrix with k lines and l columns, and element $m_{k\,l} \geq 0$ for $k \neq l$. Put simply, M would be a positive matrix if all non-diagonal elements were non negative and it would then preserve the non-negativity of the state vector. The condition $m_{k\,l} \geq 0$, k

$\neq$ l is necessary, while the stronger condition $m_{kl} > 0$, $k \neq l$ is sufficient for a nonnegative solution. Hence, starting from any nonnegative initial x_o (= price p_o), the solution (3) would remain nonnegative.

Generally, if M is a nonnegative matrix, then for some constant $b > 0$, the matrix $D = b\,I + M$ is also nonnegative and has Frobenius-Perron eigenvalue $\mu_o \geq 0$ and a corresponding positive eigenvector v_o. It follows that $\lambda_o = \mu_o - b$, ($b \in \Re_+$) is an eigenvalue of M. λ_o is real and is the eigenvalue of M with the largest negative real part; it is therefore the dominant eigenvalue of M. Two important conclusions would be drawn from this sort of transformation. That is: 1) it would be possible to translate all results of nonnegative matrices to equivalent Metzler matrices, and; 2) it would follow that λ_o is real and $x_o > 0$ such that $M\,x_o = \lambda_o\,x_o$, and for any other $\lambda \neq \lambda_o$, the $Re(\lambda) < Re(\lambda_o)$. This would guarantee a positive and stable equilibrium point for (3) without, of course, any guarantee that it would be easily observable.

At first sight, the equilibrium point of a pure exchange market economy is a unique and stable fixed-point. This finding is also responsible for a real 'déformation professionnelle' in economic thinking. For even when production with delays and time-to-build and increasing returns are added, even when endogenous money and financialization are included, or when faced with nonlinearity and myriads of interconnections (including false signaling), economists remain fixated on an inexorable unique stable fixed-point despite the warning of the Sonnenschein-Mantel-Debreu-Theorem (Sonnenschein, 1972, 1973; Mantel, 1974; Debreu, 1970, 1974). In fact, it is obvious from (3) that the equilibrium $x^* = f(x; \alpha, \omega)$ depends on the distribution of the sets α and ω, i. e., on revealed preferences and supply. Even if M is invertible, changes in budget distributions /or in the supply rate would cause x^* to wobble and to elude measurements in the state space since it would be undistinguishable from a transient point. But M is not invertible in the Walrasian system. If we were to write down the augmented matrix M and then use the Gauss-Jordan elimination to find the reduced row-echelon form of the augmented matrix of M, we would find a free variable. Positing x as the price, and knowing that the rank of M is (n-1), then the values of the (n-1) prices would depend on the value assigned to the free variable. That free variable would then be the numéraire, whose arbitrary values would yield infinite stable solutions. This is not all however. All the elements of M are functions of α and ω. Hence, as the distributions of the sets of preferences coefficients and endowments are constantly changing, even with a complete set of data at a given point in time, by the time it would take a super computer to compute $x^*(.)$, it would have already changed.

Although non observable, the stable equilibrium assumption gave substance to the Quantity Theory of Money, to Adam Smith's metaphor of the invisible hand as well as to the obsession with a stable equilibrium found in all other results popularized by Bachelier and the Chicago school. As regards the latter point, it should be noted that the deterministic wobbling motion is confused with Brownian motion; that is the first grave error. As we will show shortly, in real market economies, stable equilibria are not guaranteed; that is the second error. And both cast a serious doubt on the validity of the claims transposed *in extenso* to real markets. All that can be said is that the Walrasian construct is a reflexive and therefore controllable system; indeed, the sets of admissible controls are the distributions of α and ω. But in a perfect market each agent has only an infinitesimal influence on the control set, while collections of them are unlikely to act in unison. Hence despite the mathematical reasonableness of the stable equilibrium assumption in that incomplete model, it still cannot even be verified empirically, in particular if n is a large number. What is then the justification to carry it over to a nonlinear and complex system such as a real market economy?

2.2. A Linear Time Invariant Model

The feedback optimization procedure considered here rests on three basic concepts. That is, *multi-inputs-multi-outputs linear time invariant finite-order systems; internally stable feedback; and system norm*. In addition, there is the concept of 'well-posedness' of the optimization problem ensuring that the optimization algorithm does not break down. The aim of the optimization process is to find an LTI feedback controller that stabilizes the feedback system and minimizes the closed-loop system from the exogenous input stream to the cost of producing the output.

Consider now a market economy, E, represented by an LTI model defined by finite dimensional state space model:

$$\dot{x}(t) = A_1\,x(t) + A_2\,\omega(t) + A_3\,c(t) \quad (4)$$

$$o(t) = C_1\,x(t) + C_2\,\omega(t) + C_3\,c(t) \quad (5)$$

$$y(t) = D_1\,x(t) + D_2\,\omega(t). \quad (6)$$

Equations (4) to (6) describe an input/output economy E with an input partitioned into 2 vector components, ω (t) and c (t); ω (t) represents a set of exogenous inputs, and c (t) is both another input to E and the output of the controller K. The output of E is also partitioned into 2 vector components, o (t) representing the quadratic financial and thermodynamic costs of producing the output, while y (t) is the output to be measured and to be manipulated, which is also an input to the controller K. System E has the properties of linearity and time invariance. Linearity means that if ω_1 (t) →y_1 (t) and ω_2 (t) → y_2 (t), then $a_1 \omega_1 (t) + a_2 \omega_2 (t) \rightarrow a_1 y_1 (t) + a_2 y_2 (t)$ (a_i ε $\Re_+$). Time invariance, on the other hand, means that if ω (t) → y (t), then ω (t - τ) → y (t - τ).

The controller K is an LTI model defined by a finite dimensional state space model of the form:

$$\dot{x}_{k\,(.)} = A_k\, x_k\,(.) + B_k\, y\,(.) \tag{7}$$
$$c_k\,(.) = C_k\, x_k\,(.) + D_k\, y\,(.). \tag{8}$$

The coefficient matrices A_i, C_i and D_i in ((4)- (6)) are expected to be known, while coefficient matrices A_k, B_k, C_k and D_k are to be designed or found by the optimization algorithm.

Equation (6), the input to K, does not include the controller's output c (.). Then system (4) - (6) defines a closed-loop state space model of the form:

$$\dot{x}_{cl}\,(.) = Acl\; x_{cl}\,(.) + B_{cl}\,\omega\,(,) \tag{9}$$
$$o_{cl}\,(.) = C_{cl}\, x_{cl}\,(.) + D_{cl}\,\omega\,(.), \tag{10}$$

where

$$x_{cl} = [x_k\ \ x]^T,\ \ A_{cl} = \begin{pmatrix} A_1 + A_3\, D_k\, D_1 & A_3 C_k \\ B_k\, D_1 & A_k \end{pmatrix},\ \ B_{cl} = \begin{pmatrix} A_1 + A_3\ \ D_k\, D_1 & A_3 C_k \\ B_k\, D_1 & A_k \end{pmatrix},\ \ \begin{array}{l} C_{cl} = [C_1 + C_3\, D_2\ \ D_1\ \ C_3\, C_k] \\ D_{cl} = [C_2 + C_3\, D_k\ \ D_2]. \end{array}$$

For the controller to be stabilizing, the matrix A_{cl} must be a Hurwitz matrix.

The real-valued functions of the feedback design, specified in (9) and (10) are to be minimized with respect to the controller K, subject to the constraints of well-posedness and stabilization. The H-infinity norm is the task of minimizing the H_∞ norm $\|G\|_\infty$ of the transfer matrix G. The matrices A_i, C_i, D_i must be known but they must also be subject to the following conditions to ensure that they are suitable for the feedback optimization. That is, i) the pair (A_1, A_3) must be *stabilizable*, meaning that there exists a P matrix such that [$A_1 + A_3 P$] is a Hurwitz matrix; ii) the pair (A_1, D_1] must be *detectable*, meaning that there exists a Q matrix such that [A_1 + Q D_1] is a Hurwitz matrix; and iii) the optimization procedure must be minimizing and satisfying the condition of existence of a minimizer (not discussed).

It should be noted at this point that the input set cannot be measured accurately due its sheer size and the presence of intangible inputs for which there is no metrics, and similarly for the output set y (t). Part of the difficulty stems from the fact that x ε $\Re^n$, c ε $\Re^m$, and ω ε $\Re^q$, while the matrices A_i, C_i, D_i, P, and Q, etc., are unknown. If initial conditions and the rank of these matrices are unknown, we cannot claim to know everything about the dynamic behavior of x (t) from information from output measurement. Put more simply, stabilizability and detectability being the *sine qua non* conditions for the claim of a stable equilibrium, hence, the data requirement is too demanding to demonstrate the existence of a stable equilibrium in a real LTI market system. Furthermore, a market economy with positive and negative feedbacks is almost surely nonlinear.

3. Part II

3.1. Feedback Non-Linear H-Infinity Optimization Theory

This section examines two solution concepts in the theory of robust and optimal control of nonlinear systems based on the Hamilton-Jacobi Equations. These equations are a special case of the Hamilton-Jacobi Bellman equations representing a necessary condition describing extremal geometry in generalizing problems of the calculus of variations. The Hamilton-Jacobi inequality (HJI) plays an important role in the study of various qualitative properties of controlled dynamical systems such as stability, invariance and optimality. If a solution to a certain generalized HJI exists, then it is a sufficient condition for stability. The Hamilton-Jacobi-Isaacs equations (HJIE), on the other hand, are the nonlinear version of the Riccati equation studied in the H_∞-control problem for linear systems. We will focus on the contributions of Aliyu (2011) who summarizes all relevant topics on the subject. In particular, he shows (via state feedback H_∞-control problems for affine nonlinear systems that use the theory of dissipative systems (developed mainly by (van der Schaft, 1991; Bazar and Bernhard, 1995) that significant progress had been made. For, van der Schaft had already shown that for time-invariant affine nonlinear systems that are smooth, the state feedback H_∞-control problem is solvable by smooth feedback if there exists a smooth positive semi-definite solution to a dissipation inequality. The non-affine and affine cases considered by Aliyu will suffice for the present purpose, which is to show the necessary and formidable 'data requirement' faced by the would-be controller of the economy.

3.2. Generalities

During the 1960s and 1970s, economists were encouraged by the World Bank to build large general equilibrium models, which produced mainly insignificant results. During the 1990s onward, economists switched from linear H-infinity control developed by Zames (1981, 546-559; Francis (1987), among others, to the theory of nonlinear H-infinity control based on the efficient solution of the Hamilton-Jacobi equations (HJE) and on Hamilton-Jacobi- Bellman equations (HJBE) that extended the contributions of both Euler and Lagrange. The nonlinear case is mainly the contributions of Isidori (1997; Isidori and Astolfi (1992, pp.1283-1293) and others (Doyle, Glover, *et al*., 1989, pp.831-847; van der Schaft, 1992). In this paper, we will be guided mainly by the work of Aliyu (2011, p.133) who argues that the theory of H-infinity control becomes really useful when faced with a Hamiltonian that is independent of time. In that case, it is then possible to separate the variables in the HJE. Subsequently, it was recognized from the calculus of variation that the variational approach to problems of mechanics could equally be applied to problems of optimal control.

The H-infinity optimization problem is formulated in terms of efficient design of a stabilizing controller K (s) that minimizes the H_∞-norm of the closed-loop transfer matrix ($G_{o\,\omega}$) from the input set ω (t) to the output set o (t) for a given system E, defined by some state-space equations.

The term H_∞-control refers to the mathematical space over which the optimization takes place, which is the space of matrix-valued functions that are analytic and bounded in the open right half of the complex plane. The H_∞-norm, on the other hand, is the maximum singular value of the function over that space. The H_∞ algorithms solve suboptimal controller design problems formulated as that of finding a controller for a given $\rho > 0$ that is capable of achieving the closed-loop L_2-gain $\|G_{o\omega}\| < \rho$ if it exists.

As regards the nonlinear equivalent of the linear H_∞-control problem, van der Schaft has shown that for time-invariant affine nonlinear systems that are smooth, state feedback H_∞-control problems are solvable by smooth feedbacks if there exists a smooth positive semi-definite solution to a dissipative inequality, or equivalently, an infinite horizon HJB-inequality, which is the same as the Hamilton-Jacobi-Isaacs (HJI)-inequality found by Basar. The solution of the output-feedback problem with dynamic measurement feedback for affine nonlinear systems was achieved by Isidori (1997; Ball, Walker, *et al*.,1993, pp.546-559), among others. Most of these developments are succinctly summarized in Aliyu who has also examined in dept a series of nonlinear affine and non-affine H_∞-control problems. We will consider two of Aliyu's problems here. The first, the state feedback problem, represents the kind of problems studied by economists in the 1980s. The second arises when the states of the system are not available for feedback or when the output is used for feedback. It is then called: *Robust output measurement feedback nonlinear H_∞-control*. It is a more elaborated model in the affine category that includes uncertainty and parameter variations. It seems to be a better representation of the real market economic. We now consider the first.

3.3. The Non-affine Case

Consider a system or a market economy E with two types of inputs: ω(t) as a collection of exogenous disturbance inputs, and input c(t) (the output of the controller), which becomes the input to the actuator driving E. The main difference between ω (t) and c (t) is that the controller can manipulate c (t) but not ω (t). E has two outputs: o (t) (the cost performance output), and y (t) (the measured output); the latter is both an output of E and an input of the controller; and both outputs are to be measured and regulated.

The problem here is to find a controller K (s) for the generalized E (s) such that the infinity-norm of the transfer function relating input ω (t) to the performance output o (t) is minimized. The minimum gain is ρ^*. If the norm for an arbitrary stabilizing controller is $\rho > \rho^*$, then the E(s) is L_2-gain bounded. In control theory, a system ∑ with input ω (t) and output o (t) is said to have L_2-gain less or equal to ρ if $\forall x \,\varepsilon\, N \subseteq X$, $\exists$ k (x) ($0 < k(x) < \infty$, $k(0) = 0$) such that $\int_o^\infty \|o(t)\| \leq \rho^2 \int_o^\infty \|\omega\|^2 + k(x)$, $\forall t > 0$, $\forall \omega(t)$: $x(0) = 0$, and k (x) is a remaining part of the integrals from t to ∞. This leads to the concepts of available storage and storage function. Then ∑ has L_2-gain $\leq \rho$ if N = X. Applied to economy E, L_2-gain is a performance measure. To solve the H_∞-control problem one starts with a value of ρ and reduce it until $\rho^*$is reached.

To construct a typical state-feedback H_∞-control problem for a general class of non-affine non-linear systems, we follow Aliyu (2011,p. 131). Here, the plant problem is compared to economy E (t) with inputs ω (t) and c (t), and outputs are o (t) and y (t); and the controller K (s) represents a set of policies and technologies. Thus the nonlinear system is defined on some manifold $X \subseteq \Re^n$ containing the origin, expressed in local co-ordinates x_i, i ε n. The state-space equations are:

$$E(.) : \begin{cases} \dot{x} = F(x; \omega, c) \\ y = x \\ o = O(x, c), \quad x(t_o) = x_o, \end{cases} \tag{11}$$

where the variation of market price dp /dt is represented by $\dot{x}$, and $x(.) \subseteq X$ is the state vector. In addition, $c(.) \varepsilon C \subseteq \Re^q$ is a q-dimensional control input belonging to the set of admissible controls C; $\omega(.) \varepsilon W \subset \Re^s$ is the set of inputs to be tracked, which belong to the set of admissible disturbances; $y(.) \varepsilon \Re^n$ is the measured output of E; and $o \varepsilon \Re^r$ is the performance output to be controlled. Further, $F(x; \omega, c): X \times W \times C \rightarrow Z^*$ is the state dynamics function; $O(x, c): X \times C \rightarrow \Re^r$ is the controlled output function, and the controller to be synthesized is referred to as K (.). Finally, the functions F (.), and O (.) are assumed to be smooth C^k ($k \geq 1$) functions of their arguments, and the point x = 0 is assumed by economists of the 1980s to be the unique equilibrium point for E such that F (0,0,0) = 0, O (0,0) = 0 (see Scheinkman, 1976, pp.11-30; Boldrin and Montruccio, 1986, pp.26-39; Benhabib and Nichimura,1979, pp.421-444; Blatt, 1983).

On the assumption that O (x, c) is linearizable, the matrix $\partial O / \partial c$ has full rank l. Letting T* be the cotangent bundle of dim 2n, the Hamiltonian function for the economy E is: $H: T^* X \times W \times C \rightarrow \Re$ as :

$$H(x, l, \omega, c) = l^T F(x; \omega, c) + (1/2)\|O(x, c)\|^2 - (1/2)\rho^2\|\omega\|^2. \tag{12}$$

Equation (12) is locally concave with respect to ω and locally convex with respect to c near the origin, which is also the equilibrium point. Hence, there exists a unique saddle-point (ω, c) for each (x, l) near the origin zero. From the rank l and the Implicit Function Theorem, there exist smooth functions ω*(x, l) and c*(x, l), defined in the neighborhood of the point (0, 0) such that ω*(0, 0) = 0, c*(0, 0), satisfying:

$$\partial H(x, l, \omega^*(.), c(.)) / \partial \omega = \partial (x, l, \omega^*(.), c^*(.)) / \partial c) = 0. \tag{13}$$

Further, suppose there exists a non-negative C^1 function $Z^*: X \rightarrow \Re$, satisfying the inequality:

$$H^*(x, Z_x^T(.)) = H[(x, Z_x^T(x), \omega^*(x, Z_x^T(x), c^*(x, Z_x^T(x))] \leq 0. \tag{14}$$

Then the feedback law is $\omega^* = \omega(x, Z_x^T(.))$, $c^* = c(x, Z_x^T(.))$. Substituting $c^* = c(x, Z_x^T(.))$ in (11) yields the closed-loop system, satisfying:

$$Z_x^T(x) F(x, \omega, c^*(x, Z_x^T(x)) + (1/2)\|O(x, c^*(x, Z_x^T(x)\|^2 - (1/2)\rho^2\|\omega\|^2 \leq 0, \tag{15}$$

which is dissipative with respect to the supply rate $S(\omega, o) = (1/2)[\rho^2\|\omega\|^2 - \|o\|^2$ with storage function Z in the neighborhood of (x, ω) = (0, 0), and $\rho \varepsilon \Re_{++}$. In this case and the following one, dissipation with respect to the supply rate means that a part of input energy is dissipated in the form of heat and waste.

Obviously, in a physical system, control engineers would measure the variable (usually a unique signal) with a reasonable accuracy. In a social science, on the other hand, this task is much more difficult. For all intents and purposes, the set ω is infinite and contains intangible elements such as agents' confidence for which there is no metric. As the Hamiltonian is dissipative in conformity with the Second Law of thermodynamics, the function $Z: X \rightarrow \Re$ exists, but it and all other functions, including the optimal feedbacks ω*(.), and c*(.), are unknown. Hence, the controls cannot be synthesized to guarantee the existence of a stable equilibrium. Further, the above problem neglects important features of a real market economy. For example, what Aliyu calls 'un-modeled uncertainties' contain parameter variations (already discussed in (3)), and uncertainties arising out of the measurements of certain intangibles such as 'herd behavior', consumers' confidence, etc, that are sets in (-1, 1). Perhaps for all these reasons, the economists that ventured into optimal control never succeeded in either observing or demonstrating empirically the existence of an equilibrium point. To add more realism to (11), we consider another Aliyu's model (p.153), which is also discussed in Frieling *et al*., 1996, pp. 264-269; Anderson *et al*., 1998, pp. 1559-1593).

3.4. The Affine Case

For the more realistic affine case, consider an affine robust measurement feedback nonlinear H_∞- control economy shown in Figure 1. This time, there are 3 inputs to E: The exogenous inputs ω (t), the output of the controller c (t), and the output of the set of uncertainties d (t) that bypasses the controller. Economy E has 3 outputs: o (t); y (t) which is an input to the controller; and b, which is an input to the set of uncertainties. The state-space equations are:

$$E: \begin{cases} \dot{x} = f(x) + \Delta f(x, u, t) + G_1(x)\,\omega + [G_2(x) + \Delta G_2(x, u, t)]\,c \\ o = G_3(x) + G_4(x)\,c \\ y = [G_5(x) + \Delta G_5(x, u, t)] + G_6(x)\,\omega; \;\; x(t_o) = x_o \end{cases} \tag{16}$$

As before, the state vector is $x \varepsilon X$; $c \varepsilon C \subseteq \Re^q$, i. e. a q-dimensional controlled input belonging to the set of admissible controls; $\omega \varepsilon W \subset \Re^s \varepsilon L_2(.)$; $y \varepsilon Y \subset \Re^p$ is the measured output of E; and $o \varepsilon \Re^v$ is the cost performance output of E to be controlled. Further, $F(x, \omega, c)$: $X \times W \times C \rightarrow Z^*$ is the state dynamics function; $O(x, c)$: $X \times C \rightarrow \Re^v$ is the controlled output function. The set of parameters that are susceptible to variations over time is $u \varepsilon U \subset \Re^s$, while Δf, ΔG_2, $\Delta G_5 \varepsilon \Psi$ are unknown functions belonging to the set of admissible uncertainties. The real C^∞ functions are:

$$\begin{cases} G_1(x): X \rightarrow M_{n \times s}(X); & G_2(x): X \rightarrow M_{n \times q}(X) \\ G_3(x): X \rightarrow \Re^v; & G_4(x): X \rightarrow M_{v \times q}(X) \\ G_5(x): X \rightarrow \Re^p; & G_6(x): X \rightarrow M_{p \times s}(X). \end{cases} \tag{17}$$

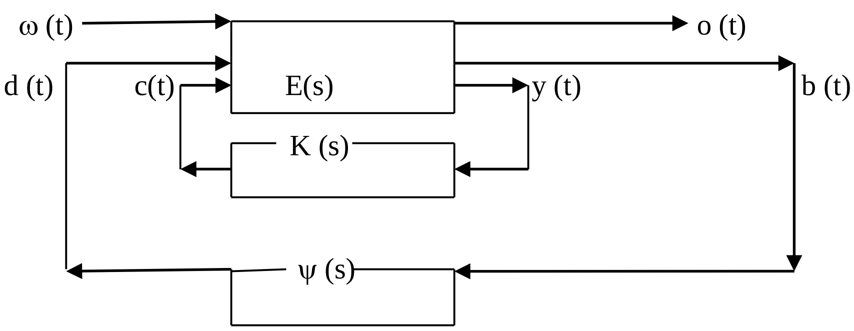

Figure 1. *Robust Measurement Feedback Nonlinear H-infinity Control Economy E.*

These are subject to the following conditions of the system matrices:

$$\begin{aligned} &\text{i) } G_3^T(.)\,G_4(.) = 0 = G_6(.)\,G_3^T(.) \\ &\text{ii) } G_4^T(.)\,G_4(.) = I = G_6(.)\,G_6^T(.), \end{aligned} \tag{18}$$

where *T* indicates the transpose operation, and I is the identity matrix. Condition i) supposes no feedback between ω(t) and o (t); condition ii) implies that the control weighting matrix is identity for the norm function o (t). It should also be specified that Δf: $X \rightarrow Z^*(x)$, where Z^* is the vector space of all C^∞ vector fields in X; $\Delta G_2(.) \rightarrow M_{n \times q}(.)$, and ΔG_5: $X \rightarrow \Re^p$.

The task now is to find a dynamic controller for E such that the closed-loop system has L_2-gain (energy) locally from the disturbance input ω (t) to output o (t) that is less or equal to some prescribed $\rho^* > 0$ with internal stability for all admissible (Δf, ΔG_2, ΔG_5) ε Ψ and for all potential parameter variations $u \varepsilon U \varepsilon \Re^s$. Aliyu has shown that to characterize Ψ some 6 additional matrices of appropriate dimensions are required. For the present purpose it suffices to say that it would be exceedingly difficult, if not impossible, to characterize Ψ in economics.

To solve the affine-robust-measurement-feedback-nonlinear-H_∞-control system, many other conditions must be satisfied, such as observability and zero-state detectability, i. e. both f, and G_3 must be locally detectable. By zero-state observable, it is meant $\exists\ \Omega \subset X$ containing $x_o = 0$ or that any trajectory starting at x_o in Ω, c (t) = 0, y (t) = 0, $\forall t \geq t_o$ implying x (t) = 0. The nonlinear system E is locally zero-state detectable if $\exists$ $N \subset X$ near x = 0 such that $\forall$ x (t_o) ε N if o (t) = 0, c (t) = 0, $\forall t \geq t_o$, implying $\lim_{t \to \infty} x(t, t_o, x_o, c) = 0$. The system is zero-state detectable if N = X. As it can be seen, there is no hope that these conditions could ever be satisfied for economy E, and there is no point discussing them further, except to say that, more importantly, there must be a smooth positive semi-definite function Z* near the origin that satisfies the Hamilton-Jacobi-Isaacs equation:

$$Z_x^*(.)\, f(x) + (1/2)\, Z_x^*(.)\, [(1/\rho^2)\, (G_1(.)\, G_1^T(.) + H_2(x)\, H_2^T(x) - G_2(x)\, G_2^T(x)]\, Z_x^{*T}(.) + (1/2)\, G_3^T(.)\, G_3(.) + (1/2)\, E_1(.)\, E_1^T(.) \leq 0, \quad (19)$$

where $H_2(.)$ and $E_1(.)$ are two of the matrices that characterize the set of admissible uncertainties Ψ.

It should be recalled at this point that our task is not to dwell into the intricacies of stabilizing a controlled economy but to show how difficult it would be to do so. Real market economies do not satisfy the properties of superposition and homogeneity due to friction, adjustment costs, cooperative and competing parts, myriads of interconnections, etc. They are obviously nonlinear and very complex. This is not to say that they are impossible to stabilize, but first optimality would have to be defined and second synthesizing policies in a rivalrous and pluralistic society would have to be found. But it should be borne in mind that, additionally, real modern markets face a measurement problem due to the lack of proper metrics. The data requirement representing myriads of interconnections is visible in the matrices $M_{n \times s}$, $M_{n \times q}$, $M_{v \times q}$, $M_{p \times s}$, and six more needed to characterize the set of uncertainties. All we know is that economy E is a nonlinear dissipative system. It is now well-known that such systems may have multiple stable equilibria, unstable equilibria and a '*strange* 'attractor'; the latter is known to have a countable set of periodic orbits of arbitrarily large period, an uncountable set of aperiodic orbits, and a dense orbit. To assert that economy E tends toward a unique and stable equilibrium on its own power when: a) x_o cannot be assumed to fall in some local stable manifold, or b) the equilibrium cannot be characterized empirically, or c) the system frequently produces undesirable outcomes, reflects "*une grave dėformation professionnelle'*.

4. Concluding Remarks

Orthodox economists are firmly attached to the idea that the economic system, by its very nature, must be a stable system even though no stable market economy has ever been observed. Yet, the notion of stable equilibrium remains the corner stone of both the 'Efficient Market Hypothesis' and the philosophy of neo-liberalism. The collapse of Western economies in 2007-2008 is an additional demonstration of the fallacy of that belief. The question now is that, as a group, economists are well versed in empirical research, why then do they hold such an inalterable belief in unobservable stable equilibria?

This paper attributes this preoccupation to three causes. That is, the Walrasian pure exchange economy; the fact that market economies, being social constructs, are theoretically controllable; and the total neglect of the analyses of complex systems. This paper shows that the Walrasian pure exchange economy, where the notion of stable equilibrium found its first mathematical expression, may be a fine exercise that is nevertheless far-removed from the complexities of areal market economies. Indeed, market economies are social constructs designed to facilitate exchange; they should, therefore, be controllable in theory. The paper then uses the new advances in affine and non-affine nonlinear feedback H-infinity control theory to show that the lack of proper metrics and the data requirements preclude all attempts at empirical verifications. Moreover, market economies are nonlinear systems subject to multiple interconnections, parameter variations, and uncertainties. Their equilibria may be multiple (as ascertained by the Sonnenschein-Mantel-Debreu Theorem), unstable, and deterministically chaotic. All depend on uncertainties and parameter values. Sensitivity to parameter variations, for example, means that minuscule changes here may produce unpredictable and huge undesirable results there. In addition, if the attractors of such systems are non-hyperbolic, then their outputs are extremely sensitive to noise. It then follows that in market economies, where information sets of participants are incomplete, observed outputs contain a noisy component that cannot be filtered out and therefore output measurements are bound to be spurious. Faced with complex systems, it is futile to attempt to establish causes and effects. Rather, it is wiser to start with policies defining the embedding space and then to look for correlates in observed and enduring patterns thrown-off by such systems.

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