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Editor's Introduction to Volume 2, Issue 3 of Expert Journal of Economics

Simona VINEREAN*

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In this third issue of the second volume of *Expert Journal of Economics*, we gathered various interesting articles exploring the effect of nonzero autocorrelation coefficients on the sampling distributions, the premises of the modern market economy, the analysis of consumer behavior in the process of search and increased wage, and the repercussions of the measurement errors in misleading results in the estimation of various population parameters. We are appreciative of the opportunity to publish such meaningful contributions to economics knowledge. Further, I present a short description of each article that is published in *Expert Journal of Economics*, vol. 2, issue 3.

In *The Effect of Nonzero Autocorrelation Coefficients on the Distributions of Durbin-Watson Test Estimator: Three Autoregressive Models*, Mei-Yu Lee (2014) discovers three interesting results for the application of a serial correlation test for an example of Durbin-Watson test estimator when the errors have nonzero autocorrelation coefficient in first-order autoregressive model. The author also compares three models to show the effect of nonzero autocorrelation coefficients on the sampling distributions of the *d* statistic.

Dominique and Rivera-Solis (2014), in their paper entitled *On Market Economies: How Controllable Constructs Become Complex* study the premises of the modern market economy from different perspectives. Firstly, the authors review the Walrasian pure exchange (WPE) model and a controllable linear time invariant (LTI) model. This research shows that the Walrasian pure exchange economy does not fully exhibit the complexities of areal market economies. Secondly, they explore two solution concepts in the theory of robust and optimal control of nonlinear systems based on the Hamilton-Jacobi Equations. The article further explores the new advances in affine and non-affine nonlinear feedback H-infinity control theory and shows that empirical verifications are difficult to achieve due to the lack of proper metrics and the data requirements.

In his paper *Willingness to Overpay for Insurance and for Consumer Credit: Search and Risk Behavior Under Price Dispersion*, Sergey Malakhov (2014) analyses consumer behavior in terms of the methodological power of relative values that appear in the process of search. The author discusses the instability of the equilibrium in the saddle point that occurs when a consumer's wage increase leads to his / her motivation to reduce the search time and to increase the quality of the goods. Further, the author elaborates on the Veblen effect and the satisficing path where consumers should take risks, and considers the economic implications of giving or family altruism. The model proposed in this research also encompasses risk aversion and optimum quantity of money. Moreover, the model also could be used for the examination of the optimal taxation.

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Ouko, Kipkoech, and Kirimi's (2014) article entitled *Effects of Measurement Errors on Population Estimates from Samples Generated from a Stratified Population through Systematic Sampling Technique* studies how the presence of measurement errors has led to misleading results in estimation of various population parameters. This paper shows how the effects of measurement errors on estimates of population total and population variance when the samples of the research are drawn using systematic sampling technique from a stratified population. The results of this research indicate that systematic errors have an effect on the accuracy of the estimates by overestimating both the population total and the population variance.

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The Effect of Nonzero Autocorrelation Coefficients on the Distributions of Durbin-Watson Test Estimator: Three Autoregressive Models

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This paper investigates the effect of the nonzero autocorrelation coefficients on the sampling distributions of the Durbin-Watson test estimator in three time-series models that have different variance-covariance matrix assumption, separately. We show that the expected values and variances of the Durbin-Watson test estimator are slightly different, but the skewed and kurtosis coefficients are considerably different among three models. The shapes of four coefficients are similar between the Durbin-Watson model and our benchmark model, but are not the same with the autoregressive model cut by one-lagged period. Second, the large sample case shows that the three models have the same expected values, however, the autoregressive model cut by one-lagged period explores different shapes of variance, skewed and kurtosis coefficients from the other two models. This implies that the large samples lead to the same expected values, 2(1 – ρ_0*), whatever the variance-covariance matrix of the errors is assumed. Finally, comparing with the two sample cases, the shape of each coefficient is almost the same, moreover, the autocorrelation coefficients are negatively related with expected values, are inverted-U related with variances, are cubic related with skewed coefficients, and are U related with kurtosis coefficients.*

Keywords*: Nonzero autocorrelation coefficient, the d statistic, serial correlation, autoregressive model, time series analysis*

JEL Classification*: C32, C15, C52*

1. Introduction

Serial correlation has the most important role in autoregressive models, which is based on the regression analysis. If the data has serial correlation, then the researchers have to pay attention to it and use the correct variance-covariance matrix for estimation and forecasting. However, Lee (2014a) indicates the reasons of the difference between the errors and the residuals in regression analysis where $\mathbf{X}^T\hat{\boldsymbol{\varepsilon}} = \mathbf{0}$ is its internal constraint for the residuals, that is also affected by the values of the independent variables. Another one important factor is degree of freedom which constraints the relationship of sample size and the number of independent variables. Lee found that $\mathbf{X}^T\hat{\boldsymbol{\varepsilon}} = \mathbf{0}$ is very important when the degree of freedom is not very large in the regression analysis, thus, the autoregressive model will have to pay considerable attention on the above factors when the researchers use the serial correlation test estimator. Lee (2014b) also discusses the Z

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test that can be used in the serial correlation test of the *d* statistic when the degree of freedom is larger than 50. In that paper, Lee investigated the effects of the factors, including the variances of the errors, the values of the independent variables, on the distributions of the *d* statistic. Therefore, we do not repeat the investigations in this paper.

Due to the internal constraint and degree of freedom, we use three models, including Durbin-Watson model (Durbin and Watson, 1950, 1951), the common autoregressive model, and autoregressive model with one-lagged period, AR(1) model, (Savin and White, 1978), to discuss the effect of nonzero autocorrelation coefficients on the distributions of the Durbin-Watson test estimator, the *d* statistic. The reason we choose the *d* statistic is that its formula is the combination of the residuals and no one researches from the viewpoint of the degrees of freedom. In fact, the Durbin-Watson model has unfixed variance-covariance matrix, and the AR(1) model is restricted in the range from -0.5 to 0.5. We intend to show the differences from the distributions of the *d* statistic among three models, and to compare the coefficients of the *d* statistic between any two models.

This paper complements and explains if the null hypothesis is the nonzero autocorrelation coefficients, H_0: $\rho = \rho_0$, $\rho_0 \neq 0$, then how the distributions of the *d* statistic will become and what are the differences among three models. We show that the three models have the same autocorrelation coefficients as the null hypothesis in the robust analysis, but in the small samples, the three models have different distributions of the *d* statistic. It is worthy noting that the importance of the null hypothesis with the nonzero autocorrelation coefficient. When the researchers can know the data of the exactly autocorrelation coefficient, they can accurately forecast and judge the future. Even the critical value table can be built without neglecting the properties of the errors and the values of the independent variables. The structure of the paper is as follows. Section 2 describes the three model settings and the simulation procedure. Section 3 explores our simulation results that have (1) the patterns of four coefficients among three models when the sample is 57 and the number of independent variables is 6, and (2) the patterns of four coefficients between any two models when the sample is 1000 and the number of independent variables is 6. Section 4 presents the conclusions and discussion of the results.

2. The model

Consider a linear regression model with k regressors and T sample sizes, as

$$\underset{(T\times 1)}{\mathbf{Y}} = \underset{(T\times k)}{\mathbf{X}}\,\underset{(k\times 1)}{\mathbf{B}} + \underset{(T\times 1)}{\mathbf{e}}$$

Each ε_t is the error matrix, **e**, and satisfied with three conditions that are

(i) ε_t is i.i.d. Normal distribution.
(ii) $E(\varepsilon_t) = 0$ and $Var(\varepsilon_t) = \sigma^2$ for all t.
(iii) $E(\varepsilon_t \times \varepsilon_{t-1}) = 0$ and $E(\varepsilon_i \times \varepsilon_j) = 0$, $|i - j| > 1$, for all t and i, j = 1, 2, …, T.

$\mathbf{Y} = \mathbf{X}\ \mathbf{B} + \mathbf{e}$ is constrained by $E(\mathbf{e}) = 0$ and $E(\mathbf{X}^T\mathbf{e}) = 0$. Use OLS and get the estimator of coefficients, $\hat{\mathbf{B}} = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{Y}$, due to the constraint of $\mathbf{X}^T\mathbf{e} = 0$. Thus the residuals are $\hat{\boldsymbol{\varepsilon}} = (\mathbf{I} - \mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T)\boldsymbol{\varepsilon}$, which is satisfied with $E(\hat{\boldsymbol{\varepsilon}}) = 0$ and $\mathbf{X}^T\hat{\boldsymbol{\varepsilon}} = 0$ and the degree of freedom being T-p-1. The sum of square residuals will be

$$E(\hat{\boldsymbol{\varepsilon}}\hat{\boldsymbol{\varepsilon}}^T) = \sigma^2(\mathbf{I} - \mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T) \quad (1)$$

The condition (iii) guarantees the errors are independent from each other. However, the serial correlation model has broken condition (iii). In order to test the extensibility of the *d* statistic in the serial correlation models, the models are that

- Model A is the Durbin-Watson model introduced by Durbin and Watson in 1950.
- Model B is the serial correlation model in common. This is also the benchmark model.
- Model C is the autoregressive error procedure with one lagged period.

2.1. Model A

The serial correlative condition of Model A is $\varepsilon_{t+1} = \rho\varepsilon_t + \mu_{t+1}$, where t = 0, 1, …, T-1, ρ is population autocorrelation coefficient of ε_{t+1} and ε_t, μ_{t+1} is i.i.d. Normal distribution with $E(\mu_{t+1}) = 0$ and $Var(\mu_{t+1}) = \sigma^2$ for all t. The specialist property of Model A is the unfixed variance of the error when t increases, that is,

$$\varepsilon_{t+1} = \rho\times\varepsilon_t + \mu_{t+1} = \sum_{j=1}^{t+1}\left(\rho^{t+1-j}\times\mu_j\right),$$

$$Var(\varepsilon_{t+1}) = \sigma^2\sum_{j=1}^{t+1}\left((\rho^2)^{t+1-j}\right)$$

$$E(\varepsilon_t\times\varepsilon_{t+1}) = \rho\times\left(\sum_{j=1}^{t}\left((\rho^2)^{t+1-j}\right)\right)\times\sigma^2,$$

$$\rho(\varepsilon_t,\varepsilon_{t+1}) = \frac{\rho\times\left(\sum_{j=1}^{t}\left((\rho^2)^{t+1-j}\right)\right)}{\sqrt{\sum_{j=1}^{t}\left((\rho^2)^{t+1-j}\times\mu_j\right)\times\left(\sum_{j=1}^{t+1}\left((\rho^2)^{t+1-j}\times\mu_j\right)\right)}},$$

where $\rho(\varepsilon_t, \varepsilon_{t+1})$ is the sample autocorrelation coefficient. If t approaches to infinite, then $Var(\varepsilon_t) = \sigma^2 / (1 - \rho^2)$, $E(\varepsilon_t \times \varepsilon_{t+1}) = \rho\, Var(\varepsilon_t)$ and $\rho(\varepsilon_t, \varepsilon_{t+1}) = \rho$ (see the proofs in Appendix I).

2.2. Model B

The serial correlation condition in Model B is $\varepsilon_{t+1} = \rho\varepsilon_t + \mu_{t+1}$, where $E(\varepsilon_t \times \varepsilon_{t+1}) = \rho\sigma^2$, t = 1, 2, …, T-1, thus, μ_{t+1} is *i.i.d.* Normal distribution with $E(\mu_{t+1}) = 0$ and $Var(\mu_{t+1}) = (1 - \rho^2)\,\sigma^2$ for all t. This serial correlation condition indicates the conditional ε_{t+1} on ε_t is Normal distribution with $E(\varepsilon_{t+1} \mid \varepsilon_t) = \rho\varepsilon_t$ and $Var(\varepsilon_{t+1} \mid \varepsilon_t) = (1 - \rho^2)\,\sigma^2$. Therefore, the variance-covariance matrix is

$$E(\varepsilon\varepsilon^T) = \sigma^2\begin{bmatrix} 1 & \rho & \rho^2 & ..... & \rho^{T-2} & \rho^{T-1} \\ \rho & 1 & \rho & ..... & \rho^{T-3} & \rho^{T-2} \\ \rho^2 & \rho & 1 & ..... & \rho^{T-4} & \rho^{T-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \rho^{T-2} & \rho^{T-3} & \rho^{T-2} & ..... & 1 & \rho \\ \rho^{T-1} & \rho^{T-2} & \rho^{T-3} & ..... & \rho & 1 \end{bmatrix},$$

The special property of Model B is

$$E(\varepsilon_t\times\varepsilon_{t+1}) = \rho\times\sigma^2,$$

$$\rho(\varepsilon_t,\varepsilon_{t+1}) = \frac{\rho\times\sigma^2}{\sqrt{\sigma^2\times\sigma^2}} = \rho\ .$$

2.3. Model C

The serial correlation condition in Model C is $\varepsilon_{t+1} = \rho\varepsilon_t + \mu_{t+1}$, where $E(\varepsilon_t \times \varepsilon_{t+1}) = \rho\sigma^2$, t = 1, 2, …, T-1, and $E(\varepsilon_t \times \varepsilon_{t+j}) = 0$, j > 1. Thus, the variance-covariance matrix is

$$E(\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}^{T}) = \begin{bmatrix} \sigma^2 & \rho\times\sigma^2 & 0 & ..... & 0 & 0 \\ \rho\times\sigma^2 & \sigma^2 & \rho\times\sigma^2 & ..... & 0 & 0 \\ 0 & \rho\times\sigma^2 & \sigma^2 & ..... & 0 & 0 \\ ..... & ..... & ..... & ..... & ..... & ..... \\ 0 & 0 & 0 & \rho\times\sigma^2 & \sigma^2 & \rho\times\sigma^2 \\ 0 & 0 & 0 & ..... & \rho\times\sigma^2 & \sigma^2 \end{bmatrix}$$

It should be noted that the autocorrelation coefficients cannot be more than 0.5 and less than -0.5, or the model would be flawed.

2.4. The Durbin-Watson test

As to $\rho = 0$, the three models become one model where the errors *i.i.d.* Normal distribution with $E(\varepsilon_t) = 0$ and $Var(\varepsilon_t) = \sigma^2$. Thus, the joint probability density function of the errors is

$$f\left(\boldsymbol{\varepsilon}_1,\cdots,\boldsymbol{\varepsilon}_T\right) = \left(\sigma\sqrt{2\pi}\right)^{-T} e^{-\left(\sum_{t=1}^{T}\varepsilon_t^2\right)\left(2\sigma^2\right)^{-1}}$$

where $-\infty < \varepsilon_t < \infty$ and t = 1, 2,…,T, and then those residuals that are calculated from the Original Least Square (OLS) method will be also restricted by the internal constraint, $\mathbf{X}^{T}\hat{\boldsymbol{\varepsilon}} = 0$. The DW test statistic is not noised by σ^2. Unfortunately, the lack of discussions of the *d* statistic is not only property of central limited theorem, that has been discussed by Lee (2014b), but also the effect of nonzero autocorrelation coefficients on the distributions of the *d* statistic which has different variance-covariance matrices in three models. As to the hypotheses, H_0: $\rho = \rho_0$ and H_1: $\rho \neq \rho_0$, the joint probability density function is

$$f\left(\varepsilon_1,\ldots,\varepsilon_T\right) = \left(\frac{1}{\sqrt{2\pi\sigma}}\right)^{T} \times e^{-\frac{\boldsymbol{\varepsilon}^{T}\boldsymbol{\Sigma}\boldsymbol{\varepsilon}}{2|\boldsymbol{\Sigma}|^{0.5}}},$$

where $\Sigma = E(\hat{\boldsymbol{\varepsilon}}\hat{\boldsymbol{\varepsilon}}^{T})$ and $-\infty < \varepsilon_t < \infty$ and t = 1, 2,…, T.

Durbin and Watson (1950, 1951) build the *d* statistic for testing the serial correlation of the data when the null hypothesis is a zero autocorrelation coefficient. The *d* statistic is

$$DW = \frac{\sum_{t=1}^{T-1}\left(\hat{\varepsilon}_t - \hat{\varepsilon}_{t+1}\right)^2}{\sum_{t=1}^{T}\hat{\varepsilon}_t^2}$$

where $\hat{\varepsilon}_t = Y_t - \hat{Y}_t$. However, the mathematical transformations of joint probability density functions, from the errors to the residuals and from the residuals to the *d* statistic, are not 1-1 relation and cannot find the jocabian functions, that is,

$$f\left(\hat{\varepsilon}_1,\ldots,\hat{\varepsilon}_{T-1}\right) = f\left(\varepsilon_1,\ldots,\varepsilon_T\right)\times\frac{\partial\left(\varepsilon_1,\ldots,\varepsilon_T\right)}{\partial\left(\hat{\varepsilon}_1,\ldots,\hat{\varepsilon}_{T-1}\right)},$$

and

$$DW = f\left(\hat{\varepsilon}_1,\ldots,\hat{\varepsilon}_{T-1}\right)\times\frac{\partial\left(\hat{\varepsilon}_1,\ldots,\hat{\varepsilon}_{T-1}\right)}{\partial\left(DW\right)}.$$

Because the exactly sampling distributions of the *d* statistic cannot be found, the computer simulation is needed whatever the autocorrelation coefficient is zero or nonzero.

3. Simulation procedure

The computations are performed in C++ on Intel Core i7 desktop. In order to control the internal constraint of regression and to derive the probability density functions of the *d* statistic, we use a new simulation process based on random number method to overcome the problems of probability transformation.† Thus, the Durbin-Watson test estimator can be simulated under null hypothesis, H_0: $\rho = \rho_0$, where $\rho_0 \neq 0$. The computer repeats 2^{16} calculations per time to get 2^{16} values of the Durbin-Watson test. The research method is as followed.

Step 1: Give the intercept and slope value, $\beta_0 = \beta_1 = \beta_2 = \ldots = \beta_k = 0$, and the data set of independent variables.

Step 2: Get the error data set of normal distribution which sample size is *T*. Here, the error value is independently.

Step 3: According to the linear regression model setting and computing the data set of dependent variable, $\mathbf{Y} = \mathbf{XB} + \boldsymbol{\varepsilon}$.

Step 4: Calculate the point-estimated values of regression coefficient and getting the estimated values of dependent variable, $\hat{\mathbf{Y}} = \mathbf{X}\hat{\mathbf{B}}$.

Step 5: Calculate the data set of residual, $\hat{\boldsymbol{\varepsilon}} = \hat{\mathbf{Y}} - \mathbf{X}\hat{\mathbf{B}}$.

Step 6: Get the value of the *d* statistic.

Every time generate 2^{16} values by repeating Step 2 to Step 6. Those values can generate a frequency table and then calculate the sampling distributions and coefficients. Because 2^{16} values per time is large enough, the sampling distributions of the *d* statistic can be viewed as population distributions. The error of coefficients between real value and estimated value is from 1/1000 to 1/10000.

When the sampling distributions of the *d* statistic are generated, the computer calculates the means, variances, skewedness, and kurtosis coefficients. The skewedness and kurtosis coefficients can ensure more whether the sampling distributions of the *d* statistic are Normal distribution or not in the three models. The paper defines the coefficients of the *d* statistic as

- ρ = X1 is the autocorrelation coefficient of the errors.
- E(DW) = X2 is the mean of the *d* statistic.
- Var(DW) = X3 is the variance of the *d* statistic.
- σ(DW) = X4 is the standard deviation of the *d* statistic.
- γ_1(DW) = X5 is the skewedness of the *d* statistic.
- γ_2(DW) = X6 is the kurtosis of the *d* statistic.

4. Simulation results

First, the computer calculation depends on the values of independent variables (Appendix II), 6 regressors, the variance and autocorrelation coefficients of the errors. The sampling distributions of the *d* statistic have four coefficients which are patterned by the autocorrelation coefficients of the errors from -0.99 to 0.99 for Model A and Model B, and form -0.49 to 0.49 for Model C, as shown in Table 1. The small sample case, T = 57, shows the effect of the autocorrelation coefficient on the coefficients of the sampling distributions of the *d* statistic.

Table 1. *The extreme values of the coefficients of in three models when the autocorrelation coefficient is nonzero (T = 57, k =6)*

		X2	X3	X4	X5	X6
Model A	Max	3.659836288	0.066942063	0.258731643	0.775942865	5.882692292
	Min	0.496834365	0.020959918	0.144775404	-1.447411589	2.897774026

† The software of Durbin-Watson test is provided by C.C.C. Ltd. The software of Durbin-Watson test model (Model B) is available online on the website: https://www.facebook.com/pages/Welsh-Corgi-Program/606775822740593. The traditional Durbin-Watson test model is based on Imhof (1960) and Pan (1968).

Model B	Max	3.680587671	0.066940095	0.258727839	0.806132468	7.191216908
	Min	0.492612374	0.01518781	0.123238833	-1.736253752	2.896743261
Model C	Max	2.807037254	0.066956001	0.258758576	0.230931679	3.039399415
	Min	1.168332061	0.039141875	0.197843058	-0.237106198	2.898300265

Table 1 illustrates the maximum and minimum of five coefficients in three models. By comparison of Model A and B, two models have the same minimum of E(DW), maximum of Var(DW) and σ(DW), but slightly different maximum of E(DW), minimum of Var(DW) and σ(DW). Moreover, Model B is more positive-skewed and centralized than Model A. However, Model C has the most extreme differences of E(DW), γ_1(DW) and γ_2(DW) than Model A and Model B, except for maximum of Var(DW) and σ(DW).

Although Table 1 shows the five coefficients of the *d* statistic, we still do not know the effect of autocorrelation coefficients on the sampling distribution of the *d* statistic. Therefore, Table 2 illustrates the plots of four coefficients where the vertical axis is E(DW), Var(DW), γ_1(DW) and γ_2(DW), separately, and the horizontal axis represents the autocorrelation coefficients. Those plots assist us to investigate whether the *d* statistic is Normal distribution and how the autocorrelation coefficients affect the sampling distribution of the *d* statistic. As to the whole range of ρ in three models, E(DW) is negatively and linearly related with ρ. This implies that dE(DW) / $d\rho < 0$. The plot of E(DW) also passes through around 2.0028 (Model A and B) and 2.0027 (Model C) as $\rho = 0$. The reason is that the negative ρ causes the errors and the residuals to fluctuate up and down from t to t + 1 period, however, the positive ρ leads to one and fixed direction for the errors and residuals. Second, Var(DW) is an inverted-U shape. This implies that the higher the |ρ| is, the lower the Var(DW) is. However, the maximum of Var(DW) occurs at $\rho = 0.02$ in Model A, at $\rho = 0.01$ in Model B and at $\rho = -0.01$ in Model C. This also shows that the maximum of Var(DW) is not at $\rho = 0$. This is because the different assumption of variance-covariance matrix. From the view of E(DW) and Var(DW), E(DW) cannot be used to derive Var(DW) because the linear relationship cannot represent the U-shape relationship, especially when the autocorrelation coefficient is nonzero.

Table 2 also illustrates the plots of the skewed and kurtosis coefficients. The plots of skewed coefficient are cubic shape which shows the higher the ρ is, the higher the skewed coefficient is. The skewed coefficients are positive when $\rho > 0$. Model C has considerable shape of skewed coefficients by comparison with Model A and Model B. Although the kurtosis coefficient is 2.8978 in Model A, 2.8975 in Model B, and 208990 in Model C when $\rho = 0$, the minimum of the kurtosis coefficient occurs in $\rho = 0.01$ in three models. If |ρ| becomes larger than 0.01, the kurtosis coefficient increases, in particular, the higher the negative ρ is, the higher the kurtosis coefficient is in Model A and Model B. The plots show that the assumptions of Model B lead to the higher kurtosis coefficient than Model A when |ρ| becomes larger and close to higher relation. We also find that the kurtosis coefficient is larger than 3 when $\rho \leq -0.37$ in three models, but occurs when $\rho > 0.39$ in Model B and Model C, and when $\rho > 0.38$ in Model A.

Thus, we can obtain the proposition as follows.

Proposition 1.

(1) dE(DW) / $d\rho < 0$. When $\rho = 0$, E(DW) = 2.00 accurate to the second decimal place.

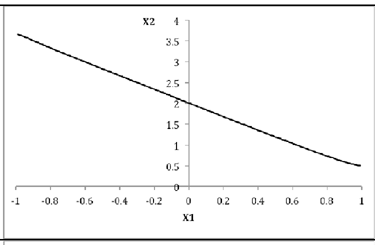
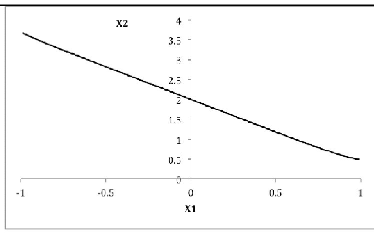
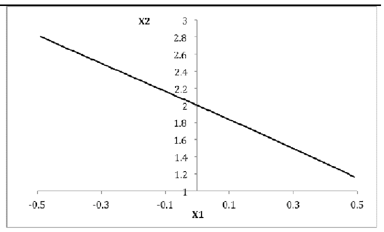
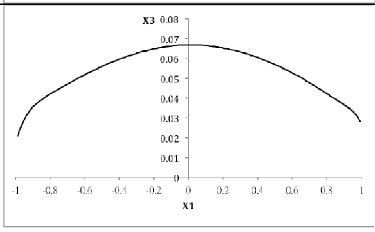
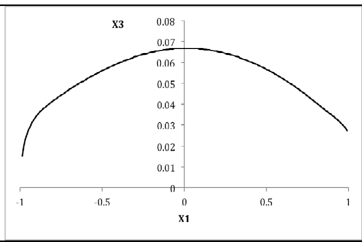
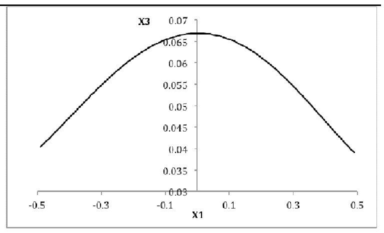
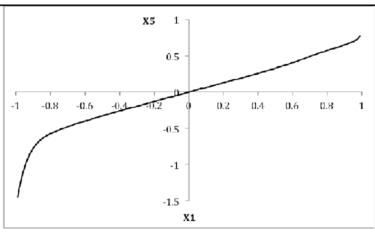
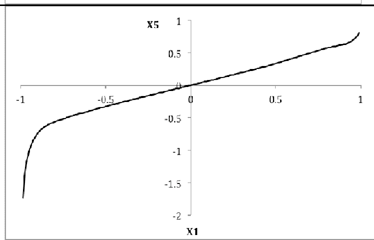
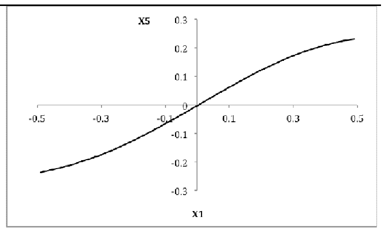
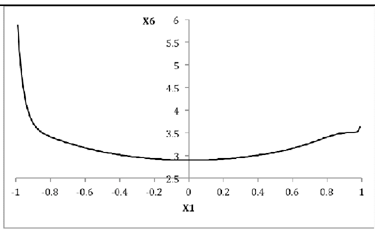
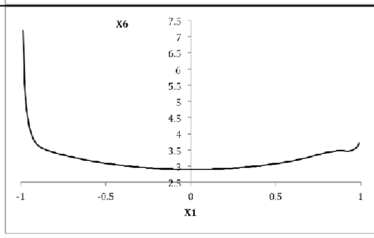
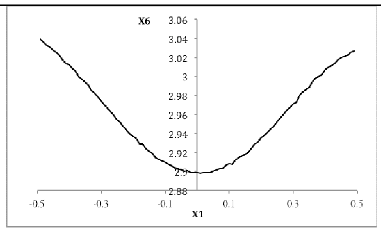
(2) When $\rho \leq 0$, dVar(DW) / $d\rho > 0$ and dVar(DW) / $d\rho < 0$ when $\rho > 0$. The second-order condition is d^2Var(DW) / $d\rho^2 < 0$.

(3) $d\gamma_1$(DW) / $d\rho > 0$. When $\rho = 0$, γ_1(DW) = -0.00 accurate to the second decimal place.

(4) When $\rho \leq 0$, $d\gamma_2$(DW) / $d\rho < 0$ and $d\gamma_2$(DW) / $d\rho > 0$ when $\rho > 0$. The second-order condition is $d^2\gamma_2$(DW) / $d\rho^2 > 0$.

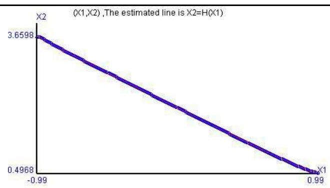
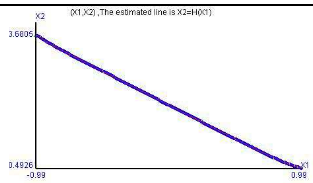
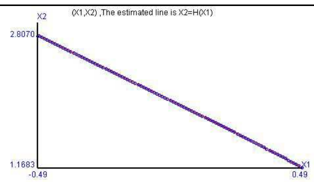
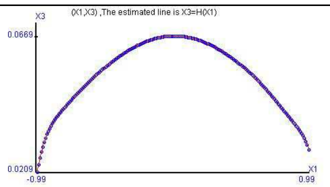
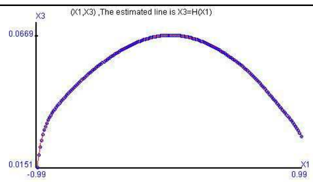
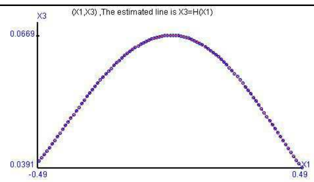
In the small sample case, the assumptions of $E(\varepsilon_t \times \varepsilon_{t+1})$ and the variance- covariance matrix leads to the differences among three models when the samples and the number of regressors are the same. This is because Model B has the fixed variance in the assumption of first-order autoregressive errors, meanwhile, Model C is based on the only one-lagged period effect on the errors. The four coefficients in three models show that the sampling distributions of the *d* statistic are not Normal distribution in the range of ρ. One reason is from Lee (2014a, 2014b), other reason is that nonzero ρ disturbs the errors, the residuals, its mathematical combination and the variance-covariance matrix, thus, the *d* statistic cannot display a Normal distribution.

Table 2. *The coefficients in three models when the autocorrelation coefficient is nonzero (T = 57, k =6)*

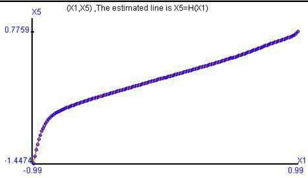
	Model A	**Model B**	**Model C**
X2			
X3			
X5			
X6			

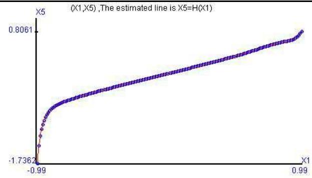
Due to the relationship of each coefficient and ρ in Table 2, we can estimate each coefficient of the *d* statistic by the autocorrelation coefficients, that is, regress each coefficient on ρ by curve-linear regression method, which is based on the Taylor expansion function, in Table 3. The horizontal axis is the values of ρ and the vertical axis is the values of each coefficient. It is noted that the three models indicate that the *d* statistic is asymmetric at 2 and has significant difference in γ_1(DW) and γ_2(DW), in particular, when the null hypothesis is H_0: $\rho = \rho_0$.

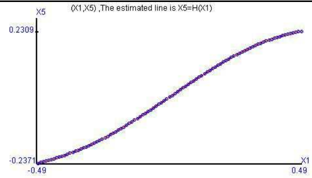
Table 3. *The estimation of each coefficient in three models when the autocorrelation coefficient is nonzero*

Symbol	**Model A**	**Model B**	**Model C**
X2			
X3			

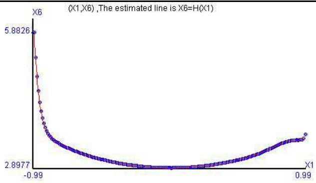
X5

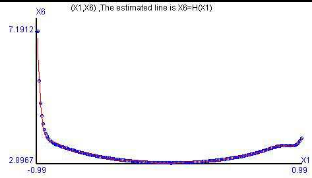


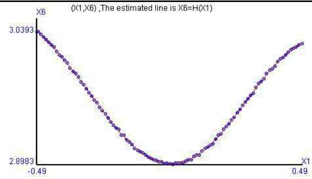




X6







The Appendix III illustrates the residual plots of each coefficient after estimation. The residual plots show that the model setting leads to the different effect of ρ on the coefficients of the *d* statistic, even the shapes of each coefficient are as similar as possible. Moreover, Model A and Model B have similar coefficients, but the residual plots are considerably different with each coefficient. The special variance-covariance matrix assumption leads to the residual plot of Model C different from others. Due to the considerable paths of coefficients as a change of autocorrelation coefficients, the *d* statistic is still sensitive and is used for hypothesis testing in three models when the null hypothesis is H_0: $\rho = \rho_0$.

4.1. Robust analysis

When the samples are large enough, the three models have the same sampling autocorrelation coefficient, $\rho(\varepsilon_t, \varepsilon_{t+1}) = \rho$, however, have different values of $E(\varepsilon_t \times \varepsilon_{t-1})$, that is,

$$E(\varepsilon_t, \varepsilon_{t+1}) = \begin{cases} \dfrac{\rho\sigma^2}{1-\rho^2} & , Model\ A \\ \rho\sigma^2 & , Model\ B, C \end{cases} .$$

Thus, the zero autocorrelation coefficient leads to $\rho(\varepsilon_t, \varepsilon_{t+1}) = \rho(\varepsilon_t, \varepsilon_{t+1}) = 0$ in the three models. If the null hypothesis is nonzero autocorrelation coefficient, H_0: $\rho = \rho_0$, and T is infinite, then

$$E(DW) = 2\,(1 - \rho_0). \qquad (2)$$

The expected values of the *d* statistic is a constant value away from 2 means that k has no impact on the robust means of the *d* statistic whatever the autocorrelation coefficient is. Furthermore, E(DW) is negatively and linear related with ρ_0 as shown in (2). However, E(DW) insufficiently represents the information of the sampling distributions of the *d* statistic when T is large enough. The second to fourth rows of Table 4 illustrate the effect of the higher moments on the sampling distributions of the *d* statistic. The second row shows that Model A and Model B have the same shape of variance, but are different from Model C. moreover, Var(DW) is affected considerably by the positive autocorrelation coefficient in Model A and by the negative autocorrelation coefficient in Model B. We also find that the higher the |ρ| is, the larger the difference between Model A and Model C (Model B and Model C) is.

The third row illustrates that the higher positive ρ leads to that (1) the skewed coefficients of Model A are larger than that of Model B and (2) there is a larger difference from the γ_1(DW) of Model A (or Model B) minus the γ_1(DW) of Model C. However, the larger negative ρ induces in the larger difference from γ_1(DW) of Model C minus γ_1(DW) of Model A (or Model B). We also can find that

$$\frac{d\gamma_1(DW)}{d\rho} \geq 0 .$$

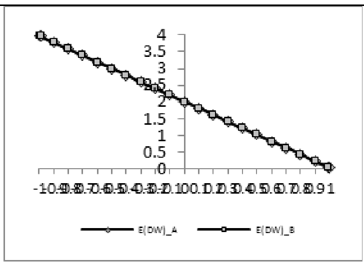
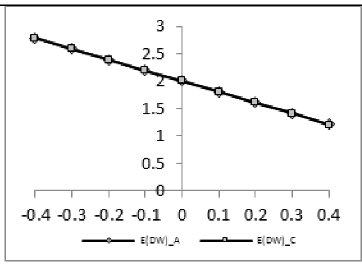
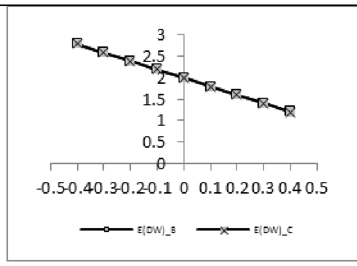
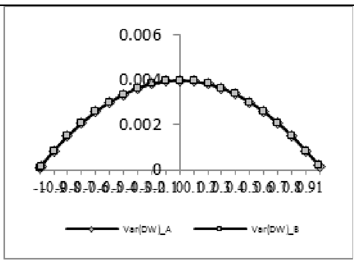
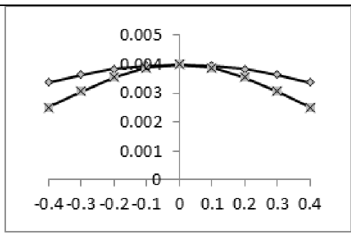
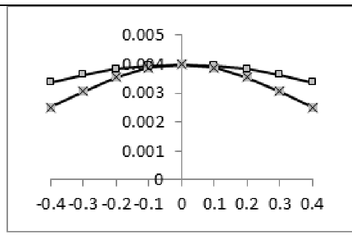
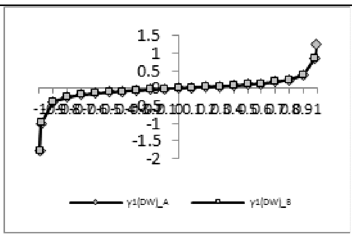
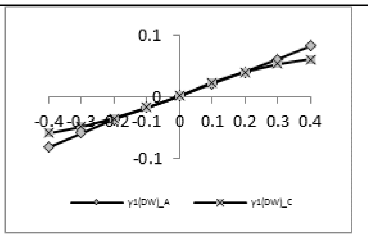
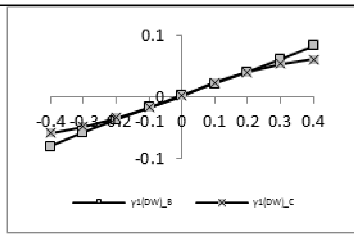
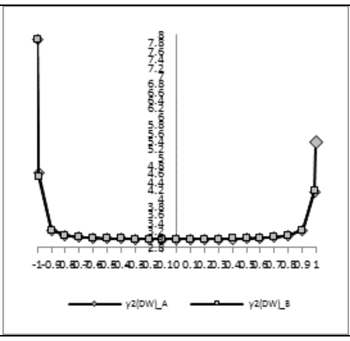
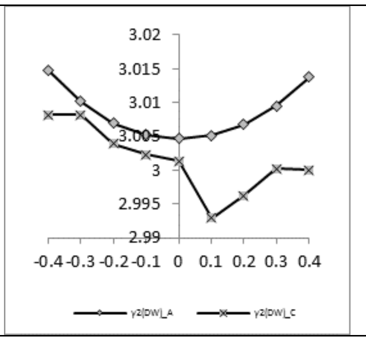
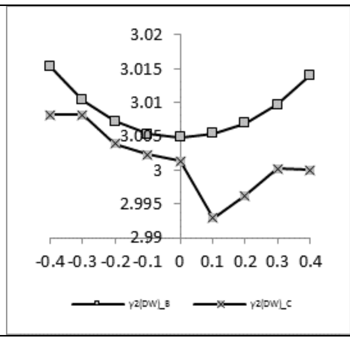
The fourth row illustrates that the smallest values of γ_2(DW) occurs in $\rho = 0$. When the $|\rho|$ becomes larger, γ_2(DW) increases in the three models, in particular, the pattern of γ_2(DW) has a kinked point at $\rho = 0.1$. The relationship between γ_2(DW) and ρ is

$$\frac{d\gamma_2(DW)}{d\rho}\begin{cases} >0 & ,if\ \rho<0 \\ <0 & ,if\ \rho>0 \end{cases},$$

and

$$\frac{d\gamma_2^2(DW)}{d\rho^2}<0.$$

***Table 4**. The comparison of the changes between three models' coefficients and autocorrelation coefficient (T =1000 and k =6)*

	Model A vs Model B	Model A vs Model C	Model B vs Model C
E(DW)			
Var(DW)			
skewness			
kurtosis			

Comparing with Table 2 and 4, the sampling distributions of the *d* statistic have the following properties.

Proposition 2. At small and large sample cases,
(1) E(DW) passes through 2 when null hypothesis is $\rho = 0$ in three models.

(2) E(DW) is negatively and linearly related with ρ in three models
(3) Var(DW) is inverted-U related with ρ in three models.
(4) Model A and B have the same shape of Var(DW), γ_1(DW) and γ_2(DW), that are different from Model C.

Table 4 also shows that the sampling distribution of the *d* statistic in three models is a Normal distribution with E(DW) = 2 due to γ_1(DW) = 0 and γ_2(DW) = 3. This is an evidence of Durbin and Watson (1950, 1951) when the samples are large enough. The second property is that the higher the positive ρ is, the more positive-skewed the sampling distributions of the *d* statistic are. The small sample case in Table 3 explains that the different variance-covariance matrix assumption affect the effect of the autocorrelation coefficient on the sampling distributions of the *d* statistic, that have different expected values and other coefficients as shown in Table 1, 2 and 3. When the samples become large, Table 4 shows the same expected value, variance and skewed coefficients among three models even through the kurtosis coefficient is a little bit different among the three models. Therefore, the large samples can eliminate the variance-covariance matrix assumption and lead the three models to become one model.

5. Conclusions

The paper runs computer simulation of serial correlation test for an example of Durbin-Watson test estimator when the errors have nonzero autocorrelation coefficient in first-order autoregressive model. We try to compare three models to show the effect of nonzero autocorrelation coefficients on the sampling distributions of the *d* statistic.

The results can be divided with three parts. The first result is from the viewpoint of the sample size. We find that whatever the sample size is, the expected values, variances, skewed and kurtosis coefficients have the same patterns of the autocorrelation coefficients in three models, separately, but part of values are not the same. We also find that the assumptions of variance-covariance matrix can be eliminated by the increasing samples, therefore, the sampling distributions of the *d* statistic have the same expected values in three models whatever the autocorrelation coefficients are. This result implies that in the long run, three time series models have the same expected values, $2(1 - \rho_0)$, that are different from the small sample case due to the expected values of Model C.

The second result is from the view of the null hypothesis with zero autocorrelation coefficient. We show that the higher the positive autocorrelation coefficient is, the lower expected values and variances of the *d* statistic are, but the higher the skewed and kurtosis coefficients are. There are reversed results in the situation of the negative autocorrelation coefficients. The third result is from the perspective of whole patterns of each coefficient. We shows that the autocorrelation coefficients are negatively and linearly related with expected values, inverted-U related with variances, cubic related with skewed coefficients and U-quadratic related with kurtosis coefficients when the autocorrelation coefficient is from the minimum to the maximum in three models whatever the samples are. The three results can supplement the literatures about the serial correlation test for an example of the sampling distributions of the *d* statistic.

6. References

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Appendix I

The model of Durbin and Watson (1950) is shown in section 2. Based on $\varepsilon_{t+1}=\rho\times\varepsilon_t+\mu_{t+1}$ and $\varepsilon_0=0$, the first error is $\varepsilon_1=\mu_1$ and then substitute into $\varepsilon_{t+1}=\rho\times\varepsilon_t+\mu_{t+1}$ and obtain the second error,

$$\varepsilon_2=\rho\times\varepsilon_1+\mu_2, E(\varepsilon_2)=0, Var(\varepsilon_2)=(1+\rho^2)\times\sigma^2.$$

Thus, $E(\varepsilon_1\times\varepsilon_2)=\rho\times\sigma^2$ and the sampling autocorrelation coefficient of the first and second errors is

$$\rho(\varepsilon_1,\varepsilon_2)=\frac{\rho}{\sqrt{(1+\rho^2)}}.$$

Following the same calculated step, we can derive that

$$\varepsilon_3=\rho\times\varepsilon_2+\mu_3=\rho^2\times\mu_1+\rho\times\mu_2+\mu_3,$$
$$E(\varepsilon_3)=0, Var(\varepsilon_2)=(1+\rho^2+\rho^4)\times\sigma^2, E(\varepsilon_2\times\varepsilon_3)=\rho\times(1+\rho^2)\times\sigma^2,$$
$$\rho(\varepsilon_2,\varepsilon_3)=\frac{(1+\rho^2)}{\sqrt{(1+\rho^2)\times(1+\rho^2+\rho^4)}},$$
$$\varepsilon_4=\rho\times\varepsilon_3+\mu_4=\rho^3\times\mu_1+\rho^2\times\mu_2+\rho\times\mu_3+\mu_4,$$
$$E(\varepsilon_3)=0, Var(\varepsilon_2)=(1+\rho^2+\rho^4+\rho^6)\times\sigma^2, E(\varepsilon_3\times\varepsilon_4)=\rho\times(1+\rho^2+\rho^4)\times\sigma^2,$$
$$\rho(\varepsilon_2,\varepsilon_3)=\frac{(1+\rho^2+\rho^4)}{\sqrt{(1+\rho^2+\rho^4)\times(1+\rho^2+\rho^4+\rho^6)}},$$

Thus, the t+1th error is that

$$\varepsilon_{t+1}=\rho\times\varepsilon_t+\mu_{t+1}=\sum_{j=1}^{t+1}\left(\rho^{t+1-j}\times\mu_j\right), t=0,1,2,....,T-1,$$
$$E(\varepsilon_{t+1})=0, Var(\varepsilon_{t+1})=\left(\sum_{j=1}^{t+1}\left((\rho^2)^{t+1-j}\right)\right)\times\sigma^2,$$
$$E(\varepsilon_t\times\varepsilon_{t+1})=\rho\times\left(\sum_{j=1}^{t}\left((\rho^2)^{t+1-j}\right)\right)\times\sigma^2,$$
$$\rho(\varepsilon_t,\varepsilon_{t+1})=\frac{\rho\times\left(\sum_{j=1}^{t}\left((\rho^2)^{t+1-j}\right)\right)}{\sqrt{\sum_{j=1}^{t}\left((\rho^2)^{t+1-j}\times\mu_j\right)\times\left(\sum_{j=1}^{t+1}\left((\rho^2)^{t+1-j}\times\mu_j\right)\right)}}$$

If t becomes infinite, then

$$Var(\varepsilon_t) = \frac{\sigma^2}{(1-\rho^2)},$$

$$E(\varepsilon_t \times \varepsilon_{t+1}) = \rho \times \frac{\sigma^2}{(1-\rho^2)},$$

$$\rho(\varepsilon_t, \varepsilon_{t+1}) = \rho$$

Appendix II

The values of independent variables are as follows.

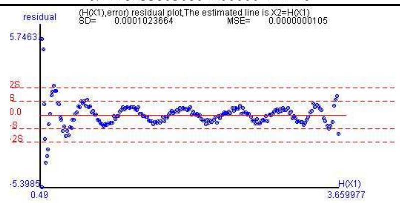
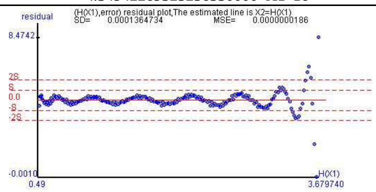
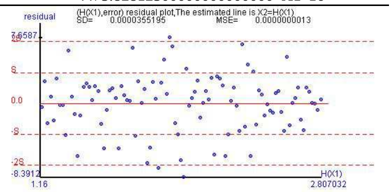
	X1	X2	X3	X4	X5	X6
1 :	8.3942796673,	5.9743248114,	10.5433072070,	4.4919705977,	3.4547532369,	0.3660935162,
2 :	11.0926182469,	8.5865059254,	6.3276410268,	2.3013060649,	1.4194860328,	1.2979850825,
3 :	11.5956287654,	4.1842520567,	7.5754211833,	8.8748009076,	9.2966283694,	9.9319530606,
4 :	5.1294293068,	-2.0229243067,	-3.7205994925,	-1.9710922902,	-2.1071922389,	-3.9631068919,
5 :	10.0930459757,	2.9087056925,	-1.2463797401,	-1.6367202069,	-1.3437181198,	-2.4391750503,
6 :	10.7591584850,	6.2449255029,	1.9952881995,	-1.5751044802,	-2.6548128155,	-1.4225324227,
7 :	11.6389242060,	9.8559733690,	19.6769166275,	20.4092287951,	21.7139250496,	21.3941172645,
8 :	8.9339369321,	9.6674110044,	6.7274054640,	6.5328385454,	5.6853699385,	5.5179365070,
9 :	9.9655867410,	3.0860552350,	3.3136013263,	1.7862885234,	2.9068953205,	2.7189989340,
10 :	13.8544611999,	10.0577880915,	3.5409810319,	-1.4431102299,	-1.8569863686,	-4.1283708796,
11 :	8.7572543061,	1.5409028398,	3.3691779308,	3.6215888116,	4.1015402009,	1.6426980000,
12 :	11.4333746774,	8.2238421572,	3.8054267172,	7.5846859124,	7.0073612155,	6.0419247516,
13 :	7.7931279457,	9.7999622586,	13.7500226003,	18.0159218476,	18.5204442419,	19.6681100550,
14 :	10.2407183608,	12.0302431166,	21.5510469818,	24.0161513894,	23.9080548620,	22.8212635916,
15 :	14.3323237799,	9.4305408181,	10.6476934619,	10.1736825488,	11.1187329437,	11.5326224000,
16 :	9.2031646401,	10.3699880329,	13.4563031661,	17.1566859980,	17.1761252852,	18.7154800011,
17 :	9.6889534618,	18.4917329222,	20.2756938987,	21.1204395704,	19.5081788708,	19.1797324594,
18 :	11.7050955005,	9.2397083531,	15.1044690444,	14.1883861443,	12.3015307628,	15.4676571064,
19 :	9.1971260006,	9.7093882312,	8.0743237967,	10.1626439416,	9.9055454481,	7.9878593242,
20 :	11.6707749148,	10.4567343750,	5.4346885708,	-0.9926995759,	-0.2580358331,	-0.7226713447,
21 :	11.0084985964,	12.7098983514,	6.8381258406,	4.2922825990,	5.9724619693,	9.0484380444,
22 :	10.4393336651,	20.4063934575,	28.1712423189,	25.2106896922,	27.0691680197,	24.3148963415,
23 :	9.2754712315,	8.4469984069,	11.1884566405,	16.2975452059,	15.6140416028,	16.8164249462,
24 :	11.0956560003,	19.5079769882,	25.0650104995,	33.9560034171,	33.8813717485,	31.1591859967,
25 :	10.0791634122,	16.6823583643,	11.4975529785,	3.6802821135,	3.6446917006,	-1.5133543254,
26 :	8.3487204240,	7.3381429614,	6.1028497337,	10.1956370626,	7.8041414071,	6.1380528833,
27 :	6.8463525509,	8.3279194586,	13.8199774185,	12.8437508964,	10.8282376376,	8.8860080326,
28 :	10.2020988906,	27.1915803537,	27.5917133148,	29.9451507344,	28.9996646890,	29.0885852288,
29 :	11.1117713869,	11.2269060716,	3.5639405330,	-3.5779110472,	-2.6505471172,	-2.4333384008,
30 :	12.0664318754,	16.8006310885,	12.7379497272,	13.1094040242,	14.5127143598,	19.0237276954,
31 :	14.1141812980,	14.8284107479,	19.3868646899,	16.6036456738,	16.3620462244,	15.8977919007,
32 :	14.4138413029,	19.1844447867,	24.2918371048,	29.4721002588,	30.1116915480,	32.1152744433,
33 :	13.3645790169,	9.4110987477,	11.5351840149,	16.2980802028,	16.3960191979,	10.9799564480,
34 :	9.5655683954,	4.8458408500,	-0.5970340237,	-5.1658276064,	-3.2146578374,	-1.0457040390,
35 :	11.5832033625,	12.2463389796,	16.8335732809,	12.3957409976,	12.6831959365,	11.3378073349,
36 :	9.6500026201,	18.1501333618,	20.0768361023,	26.4200908540,	26.6705510566,	26.4265966911,
37 :	12.3330653338,	13.0854669917,	10.3700300441,	3.1991677444,	3.3954099265,	2.7978941630,
38 :	8.6508259080,	7.4336577229,	3.3471939688,	-5.6158892589,	-5.7818123740,	-9.0915191233,
39 :	4.5136304186,	12.2936868514,	11.2979665482,	-4.2015056613,	-4.3902967105,	0.3980143565,
40 :	9.8912177478,	8.4305240403,	5.5554726620,	2.9828278842,	2.9390925524,	2.0756327884,
41 :	10.5036288232,	7.2971005321,	6.5609778313,	4.1186668666,	6.6299078094,	5.0101628919,
42 :	11.5565815704,	8.4932003482,	13.4741026052,	12.2157823319,	12.5357982707,	13.2931521944,
43 :	9.0099119726,	8.7630513853,	8.6233918610,	21.5107027433,	21.2282336352,	21.0894539511,
44 :	7.5743427980,	14.2064111846,	9.6177663912,	14.9992214910,	14.1796342689,	13.9101853988,
45 :	11.6365114336,	6.9482916849,	9.2461095245,	10.7606715127,	13.7441068642,	12.3564493977,
46 :	11.1134974482,	13.0029282291,	17.7683519485,	10.2323919463,	9.4664227563,	10.6690435813,
47 :	10.3679908394,	7.5178716331,	6.9816044043,	5.0161344127,	4.0434035535,	6.9792018302,
48 :	7.9550202448,	7.9756557376,	16.6152281740,	12.9908386728,	12.8983318498,	13.2096870593,
49 :	9.8991655726,	10.0250753674,	1.0786738992,	10.1037433773,	10.7185910088,	12.5231682908,
50 :	13.3109816579,	13.0867379409,	16.2703335687,	6.9847587587,	7.2106934802,	6.7330443978,
51 :	10.6629372996,	20.0562925451,	18.1713161972,	25.4100266454,	26.7295197714,	30.9814884118,
52 :	9.8145247150,	6.1691472249,	10.8781016015,	15.9769454371,	15.0685888697,	13.9958726602,
53 :	9.4802865631,	12.5350409185,	13.4437574182,	12.5934461630,	14.2171197996,	15.6588514033,
54 :	10.6029033874,	15.0919249001,	17.3438740646,	19.9795003428,	20.9217770362,	24.7741874651,
55 :	11.7298865621,	10.1802191618,	10.5720358408,	13.6541262143,	13.0717268281,	13.3738621297,
56 :	11.6520524994,	13.4732923593,	15.1137069458,	19.6391088533,	19.8817273292,	22.6044071700,
57 :	8.7944443244,	7.7765307842,	13.3726243393,	0.4885322455,	1.5605716470,	1.3088040036,

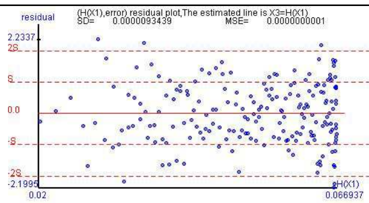
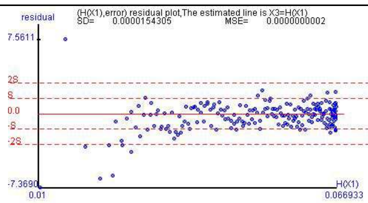
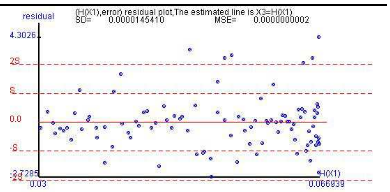
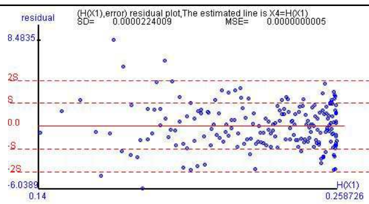
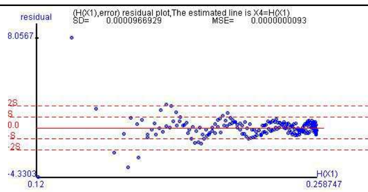
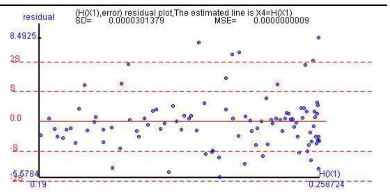
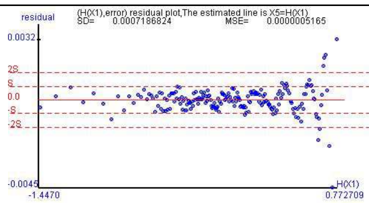
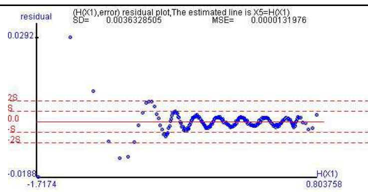
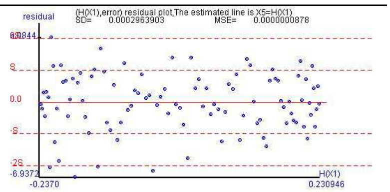
independept sample correlation coefficient------------	
r(X1,X2)=	0.2859050169
r(X1,X3)=	0.2312889512
r(X1,X4)=	0.2005213829
r(X1,X5)=	0.2265887605
r(X1,X6)=	0.2167931260
r(X2,X3)=	0.7866728319
r(X2,X4)=	0.6583522828
r(X2,X5)=	0.6616447747
r(X2,X6)=	0.6699690814
r(X3,X4)=	0.8279580562
r(X3,X5)=	0.8262028589
r(X3,X6)=	0.8057327110
r(X4,X5)=	0.9937831984
r(X4,X6)=	0.9677570153
r(X5,X6)=	0.9770514117

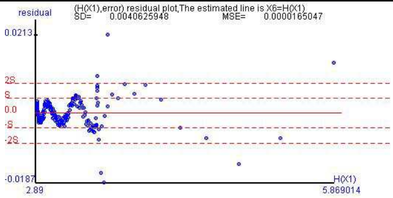
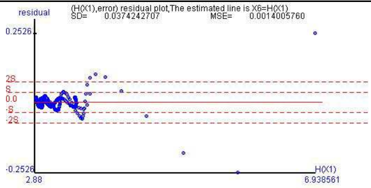
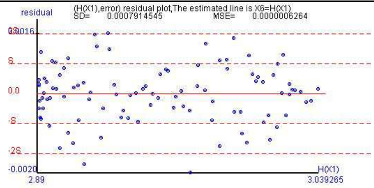
Appendix III

Table 3 shows the estimated line where we regress each coefficient of the *d* statistic on the autocorrelation coefficients in the autoregressive models with T = 57, k = 6 and three variance-covariance matrices. Appendix III shows the estimated function of each coefficient and the corresponding residual plot.

Table A-III. *The estimated function and residual plot of each coefficient in three models*

	Model A	Model B	Model C
Estimated function of X2	X2= 2.00276415438656840000+ -1.64527441226274250000*X1^1+ 0.02402413310483098000*X1^2+ 0.04011142093775976700*X1^3+ -0.19569147005677223000*X1^4+ -0.07628797340541204600*X1^5+ 1.96844828128814700000*X1^6+ 0.47423921976266570000*X1^7+ -9.18280923366546630000*X1^8+ -1.92248008535207760000*X1^9+ 23.28459286689758300000*X1^10+ 4.14671876727263110000*X1^11+ -32.32774972915649400000*X1^12+ -4.30549033957667860000*X1^13+ 23.28432893753051800000*X1^14+ 1.69847563160714190000*X1^15+ -6.77781558036804200000*X1^16+	X2= 2.00285458297548760000+ -1.64470797005871040000*X1^1+ -0.00454932160209864380*X1^2+ 0.00927235203368426130*X1^3+ 0.16936979815363884000*X1^4+ 0.28804339144714852000*X1^5+ -1.49353519082069400000*X1^6+ -1.72851022418154090000*X1^7+ 6.88667500019073490000*X1^8+ 4.85288819303698920000*X1^9+ -16.79043436050415000000*X1^10+ -6.98888725740226850000*X1^11+ 22.59680891036987300000*X1^12+ 5.07867236837829240000*X1^13+ -15.62164354324340800000*X1^14+ -1.47609063876583240000*X1^15+ 4.34942269325256350000*X1^16+	X2= 2.00281731306768050000+ -1.64539365254915730000*X1^1+ -0.06201925780624151200*X1^2+ -0.09805036787599519200*X1^3+ 0.04113501310348510700*X1^4+ -0.18425352406529782000*X1^5+ 0.23702049255371094000*X1^6+ 0.77233835506740434000*X1^7+ -21.15847778320312500000*X1^8+ 3.66662154669757000000*X1^9+ 270.37036132812500000000*X1^10+ -39.18330250936560300000*X1^11+ -1543.53515625000000000000*X1^12+ 89.36592935863882300000*X1^13+ 4213.85546875000000000000*X1^14+ -28.53425410110503400000*X1^15+ -4475.82812500000000000000*X1^16+
X2			
Estimated function of X3	X3= 0.06693278714515330000+ 0.00096872929498204030*X1^1+ -0.04520157202942237700*X1^2+ -0.00275858333043288440*X1^3+ 0.02098693347819846600*X1^4+ 0.05889438558369875000*X1^5+ -0.04170386898277911300*X1^6+ -0.55974376387894154000*X1^7+ 0.10307001079297606000*X1^8+ 2.80556001514196400000*X1^9+ -0.19054156384027010000*X1^10+ -8.35619109869003300000*X1^11+ 0.10881430686413296000*X1^12+ 15.06040996313095100000*X1^13+ 0.19172376901815369000*X1^14+ -16.10446360707283000000*X1^15+ -0.27125440162717496000*X1^16+ 9.41034364700317380000*X1^17+ 0.07943948994756056000*X1^18+ -2.30899981409311290000*X1^19+	X3= 0.06692745159035373700+ 0.00100926802133471940*X1^1+ -0.04403440576186312700*X1^2+ -0.00818694464396685360*X1^3+ -0.02190339320600287500*X1^4+ 0.17682791128754616000*X1^5+ 0.56392564276029589000*X1^6+ -2.03245097398757930000*X1^7+ -4.36716125474777070000*X1^8+ 12.71202158927917500000*X1^9+ 18.62126043066382400000*X1^10+ -47.70089125633239700000*X1^11+ -47.93447231641039300000*X1^12+ 111.62678289413452000000*X1^13+ 75.95208441745489800000*X1^14+ -164.07429504394531000000*X1^15+ -72.36105220322497200000*X1^16+ 147.18862628936768000000*X1^17+ 38.04211827833205500000*X1^18+ -73.61198830604553200000*X1^19+ -8.50006912779645060000*X1^20+ 15.73104691505432100000*X1^21+	X3= 0.06693905433940017200+ 0.00112085084241542180*X1^1+ -0.14931126567535102000*X1^2+ -0.05720452920922980400*X1^3+ 0.23701512813568115000*X1^4+ 2.79059713566675780000*X1^5+ -5.54301834106445310000*X1^6+ -82.39533479511737800000*X1^7+ 162.23004150390625000000*X1^8+ 1337.14645475149150000000*X1^9+ -2696.40917968750000000000*X1^10+ -12839.57588338851900000000*X1^11+ 26713.79687500000000000000*X1^12+ 75017.66387939453100000000*X1^13+ -160488.68750000000000000000*X1^14+ -261974.91619873047000000000*X1^15+ 573819.75000000000000000000*X1^16+ 502634.39389038086000000000*X1^17+ -1124086.00000000000000000000*X1^18+ -407490.87664794922000000000*X1^19+ 929648.50000000000000000000*X1^20+

X3			
Estimated function of X4	X4= 0.25871634963780865000+ 0.0019048159238082008 0*X1^1+ -0.08788307533168926400*X1^2+ -0.00703720701858401300*X1^3+ 0.04243934395884707600*X1^4+ 0.16138683911412954000*X1^5+ -0.27570165432905469000*X1^6+ -1.52017983049154280000*X1^7+ 1.41675009857317490000*X1^8+ 7.59491324424743650000*X1^9+ -4.54018481299863200000*X1^10+ -22.45107614994049100000*X1^11+ 8.53803876909842070000*X1^12+ 40.04951357841491700000*X1^13+ -9.34289753378470780000*X1^14+ -42.32602667808532700000*X1^15+ 5.65262481257468610000*X1^16+ 24.40732628107070900000*X1^17+ -1.51321717932046340000*X1^18+ -5.89714848995208740000*X1^19+	X4= 0.25874558828494687000+ 0.00116615943989017980*X1^1+ -0.09370305367139275500*X1^2+ 0.04326088697416707900*X1^3+ 0.22944046422412612000*X1^4+ -0.93365782871842384000*X1^5+ -2.57513941204121010000*X1^6+ 8.71797290444374080000*X1^7+ 15.23820672201236400000*X1^8+ -43.85258543491363500000*X1^9+ -50.98315056676892700000*X1^10+ 128.42374897003174000000*X1^11+ 99.67723387047044500000*X1^12+ -225.93347322940826000000*X1^13+ -113.13078427802429000000*X1^14+ 235.13924753665924000000*X1^15+ 69.18502820802859800000*X1^16+ -133.33349180221558000000*X1^17+ -17.67463392936110700000*X1^18+ 31.75831666588783300000*X1^19+	X4= 0.25872414294541740000+ 0.00218952529007765410*X1^1+ -0.28711437892281433000*X1^2+ -0.11474610300501809000*X1^3+ 0.09467011441657291500*X1^4+ 5.82031850516796110000*X1^5+ 0.56867189035256160000*X1^6+ -174.21119821071625000000*X1^7+ 4.15725120907882230000*X1^8+ 2847.46576309204100000000*X1^9+ -278.86300692148507000000*X1^10+ -27512.49142456054700000000*X1^11+ 3717.06452841684220000000*X1^12+ 161644.86083984375000000000*X1^13+ -21917.94077126681800000000*X1^14+ -567304.79101562500000000000*X1^15+ 61310.82944253087000000000*X1^16+ 1093264.07617187500000000000*X1^17+ -66382.15657681226700000000*X1^18+ -889763.74609375000000000000*X1^19+
X4			
Estimated function of X5	X5= -0.00291216222649381960+ 0.64867459781227588000*X1^1+ 0.04365262278588488700*X1^2+ -0.23647712157747591000*X1^3+ -1.35425623692572120000*X1^4+ 3.78865170210141860000*X1^5+ 16.48487353324890100000*X1^6+ -25.01537586234030600000*X1^7+ -98.04673397541046100000*X1^8+ 92.79775447481188200000*X1^9+ 326.88679599761963000000*X1^10+ -192.81265069143592000000*X1^11+ -636.72084188461304000000*X1^12+ 224.12265730113359000000*X1^13+ 718.24755907058716000000*X1^14+ -136.98606110438686000000*X1^15+ -434.69468307495117000000*X1^16+ 34.89437117650823000000*X1^17+ 108.77539741992950000000*X1^18+	X5= -0.00156830864398216360+ 0.66737813741740326000*X1^1+ -0.20871253625955433000*X1^2+ -1.32277924744723660000*X1^3+ 6.79747200198471550000*X1^4+ 22.01213340540005000000*X1^5+ -81.69792529940605200000*X1^6+ -160.22442125098451000000*X1^7+ 489.64396274089813000000*X1^8+ 620.90964811027334000000*X1^9+ -1633.90411281585690000000*X1^10+ -1355.63317866927900000000*X1^11+ 3180.45693492889400000000*X1^12+ 1675.05565517320060000000*X1^13+ -3588.04350376129150000000*X1^14+ -1094.70251322849530000000*X1^15+ 2172.19306564331050000000*X1^16+ 294.72894110489710000000*X1^17+ -545.88428068161011000000*X1^18+	X5= -0.00286740150323794300+ 0.64842448632645633000*X1^1+ 0.06563797454873565600*X1^2+ -1.23104680830800820000*X1^3+ -5.01435196027159690000*X1^4+ 26.79619901393743900000*X1^5+ 285.35144710540771000000*X1^6+ -726.24347742414102000000*X1^7+ -8435.74569702148440000000*X1^8+ 11439.42840493843000000000*X1^9+ 139541.98205566406000000000*X1^10+ -112706.41101063927000000000*X1^11+ -1372194.00341796880000000000*X1^12+ 701381.51682734489000000000*X1^13+ 8199799.37890625000000000000*X1^14+ -2657743.19020271300000000000*X1^15+ -29211744.45312500000000000000*X1^16+ 5565915.56249618530000000000*X1^17+ 57045139.81250000000000000000*X1^18+ -4919005.74142837520000000000*X1^19+ -47001062.00000000000000000000*X1^20+
X5			
Estimated function of X6	X6= 2.89526127138450340000+ -0.00659959714539581910*X1^1+ 1.05570823063877750000*X1^2+ -0.18524028745014220000*X1^3+ -6.31489422979119030000*X1^4+ 2.72991794068366290000*X1^5+ 46.29017527471736300000*X1^6+ -19.13156626373529400000*X1^7+ -163.80979793266837000000*X1^8+ 67.80064900219440500000*X1^9+ 302.79609119648160000000*X1^10+ -125.60299882292747000000*X1^11+ -279.37539797732717000000*X1^12+ 118.05200903117657000000*X1^13+ 101.57023962263294000000*X1^14+ -45.09170855581760400000*X1^15+	X6= 2.88418800220961820000+ 0.16979065437044483000*X1^1+ 2.41716662746390610000*X1^2+ -8.24893121444620190000*X1^3+ -33.18561544813537300000*X1^4+ 107.87552714906633000000*X1^5+ 241.93556902189965000000*X1^6+ -614.07626396417618000000*X1^7+ -832.88981423954260000000*X1^8+ 1776.00263896584510000000*X1^9+ 1465.67227544617090000000*X1^10+ -2725.72459515929220000000*X1^11+ -1272.83054430824970000000*X1^12+ 2115.37847287952900000000*X1^13+ 432.10185412653846000000*X1^14+ -653.76943070068955000000*X1^15+	X6= 2.89872152014899020000+ -0.01385549449634027000*X1^1+ 0.97876272603032277000*X1^2+ 0.36579956805508118000*X1^3+ -0.99048284207492276000*X1^4+ -4.57714397413656120000*X1^5+ -10.86867282170240000000*X1^6+ -14.21063726395368600000*X1^7+ 5.41359898325275250000*X1^8+ 808.61749762296677000000*X1^9+ 521.00385404010342000000*X1^10+ -7131.31718730926510000000*X1^11+ -2755.07192750513650000000*X1^12+ 25882.72533130645800000000*X1^13+ 4364.86658143642130000000*X1^14+ -34396.09129142761200000000*X1^15+

X6			

residual
(H(X1),error) residual plot,The estimated line is X6=H(X1)
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2S
S
0.0
-S
-2S
-0.0187
2.89
H(X1)
5.869014

residual
(H(X1),error) residual plot,The estimated line is X6=H(X1)
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0.2526
2S
S
0.0
-S
-2S
-0.2526
2.88
H(X1)
6.938561

residual
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S
0.0
-S
-2S
-0.0020
2.89
H(X1)
3.039265

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On Market Economies: How Controllable Constructs Become Complex

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Since Lėon Walras neoclassical economists hold an inalterable belief in a unique and stable equilibrium for the economic system which however remains to this day unobservable. Yet that belief is the corner stone of other theories such as the 'Efficient Market Hypothesis' as well as the philosophy of neo-liberalism, whose outcomes are also shown to be flawed by recent events. A modern market economy is obviously an input/output nonlinear controllable construct. However, this paper examines four such models of increasing complexity, including the affine nonlinear feedback H_∞ control, to show that the 'data requirement' precludes all attempts at the empirical verification of the existence of a stable equilibrium. If equilibria of complex nonlinear deterministic systems are most likely unstable, multiple or deterministically chaotic depending on their parameter values and uncertainties, then society should impose limits on the state space and focus on endurable patterns thrown-off by such systems.

Keywords: *Equilibrium, nonlinearity, controllability, nonlinear feedback, H_∞-control, complexity.*

JEL Classification: *C61, C62, C68, C68, D57, D58.*

1. Introduction

The basic assumptions of neo-classical economics are well-known, but this paper is mainly concerned with a subset of these. For example, it is assumed that individuals and firms *optimize* under constraints; that agents are *rational* and always have *rational expectations*; that *the more connected are networks of individual participants the less risky, stable and robust is the economic system,* etc. Undoubtedly, the most misleading of these assumptions, i.e., the one which is our main focus here, is the *claim that market economies tend toward stable equilibria* (or an optimal Pareto state of balance). It is understood that such systems may be found away from their equilibrium points as a result of exogenous shocks, but they will inexorably return to their equilibrium on their own power. As a consequence, therefore, policy gurus of neo-liberalism propagate other beliefs to the effect that markets should be allowed to make all the major economic, social, and political decisions; that the state should refrain from any attempt to control markets, or that even public institutions should be handled over to corporations for the sake of greater efficiency, etc.

Strangely enough, after more than a century and a half not a single one of these assumptions has found empirical support. Beside the observational judgment to the effect that economic agents are generally self-interested and have strong monotone preference, most of those assumptions are wishful pronouncements. Yet,

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the equilibrium assumption, for example, though unobservable, is the foundations of both the 'Efficient market Hypothesis' and neo-liberalism that have had and continue to have very undesirable outcomes, in the Western economies in particular. These counterfactual results and the inability to sustain such a pivotal assumption do not bode well for that social science. It would, therefore, be useful to reexamine a few models of market economies so as to see why the stable equilibrium assumption has always escaped empirical verification regardless of assumed models' characteristics.

This paper consists of two parts. The first reexamines two linear models. That is the Walrasian pure exchange model, and a controllable linear time invariant model. The former is incomplete but is simple to analyze; its merit lies in the fact that it provides the first mathematical expression of the stable equilibrium assumption. The latter stumbles on observability and linearity. More realistically, it is proper to suppose at the outset that a modern market is a complex construct designed to facilitate exchange, which is more natural. It should therefore be controllable. Hence, Part II appeals to the recent but well characterized L_2- gain analysis of nonlinear systems and nonlinear feedback H-infinity control to examine two classes of nonlinear models, one in the non-affine category and the other in the affine categories. All four cases show that the equilibrium of market economies may well exist in theory, but will remain forever unobservable due to the *complexity* of markets and /or due to the formidable *data requirement* for such an endeavor. In the concluding remarks, we will then offer a few suggestions on how to navigate in complex systems.

2. Part I

In this section, we will review the Walrasian pure exchange (WPE) model and a controllable linear time invariant (LTI) model. We will show, on the one hand, that the equilibrium of the WPE model can easily be inferred but not easily demonstrated empirically. Though naïve and unrealistic, it provides nevertheless the justification for a set of beliefs that may still be blocking progress in the development of the neo-classical theory of economics. The LTI model, on the other hand, completely shunts the unavoidable complexity of real markets. Despite the unrealism of both models, however, their analyses are still worthwhile, for they clearly show that their data requirement precludes all attempts to characterize a stable equilibrium.

2.1. The Beginning

The model conceived by Walras from observing the functioning of the 'Bourse de Paris' is that of a pure exchange economy. It supposes there are i consumers ($i \varepsilon m$) of j goods ($j \varepsilon n$). Each consumer devotes a fraction α^i_j of his or her budget (B) to good j such that $\Sigma_j\alpha^i_j = 1$. The budget of i comes from the sale of endowments ω^i_j such that the demand of i for good j is $x^i_j = \alpha^i_j (B) /p_j$, where p_j is the price of j. Walras supposed a one period market. Hence in the neighborhood of the equilibrium point, we have a first-order linear differential equation:

$$\begin{aligned}\dot{x} &= dg\,(1/x_j)\,[A - dg\,(\Sigma_j\omega_j)]\,x \qquad (1)\\ &= dg\,(1/x_j)\,[M]\,x,\end{aligned}$$

where $x \varepsilon X \varepsilon \Re^n$ is the state vector, and $[A - dg\,(\Sigma\omega_j)] = M_{n\,x\,n}$. For the derivation of (1), (see Dominique, 2008).

Equation (1) is an input/output construct (with inputs $\Sigma_j\,\omega_j$ and output $x \varepsilon X \varepsilon \Re^n$) driven by incentives to minimize excess demand of all goods j.

For a solution, Walras posited a tâtonnement process controlled by an auctioneer. Had he taught of an exogenous supply rate for a sequence market, (1) would have been written as,

$$\dot{x} = M\,x,\;\; x \varepsilon X \varepsilon \Re^n_+ ,\, x\,(0) = x_o > 0 \text{ as initial condition,} \qquad (2)$$

and (2) would have been represented by a linear system of differential equations whose solution is:

$$x\,(t) = e^{M\,t}\,x_{o,} \qquad (3)$$

where $e^{M\,t}$ is an n x n matrix function defined by its Taylor series, provided of course that M is invertible.

If M were a real invertible matrix of order n x n, it would be called a Metzler matrix with k lines and l columns, and element $m_{k\,l} \geq 0$ for $k \neq l$. Put simply, M would be a positive matrix if all non-diagonal elements were non negative and it would then preserve the non-negativity of the state vector. The condition $m_{k\,l} \geq 0$, k

$\neq l$ is necessary, while the stronger condition $m_{kl} > 0$, $k \neq l$ is sufficient for a nonnegative solution. Hence, starting from any nonnegative initial x_o (= price p_o), the solution (3) would remain nonnegative.

Generally, if M is a nonnegative matrix, then for some constant $b > 0$, the matrix $D = b\,I + M$ is also nonnegative and has Frobenius-Perron eigenvalue $\mu_o \geq 0$ and a corresponding positive eigenvector v_o. It follows that $\lambda_o = \mu_o - b$, ($b \in \Re_+$) is an eigenvalue of M. λ_o is real and is the eigenvalue of M with the largest negative real part; it is therefore the dominant eigenvalue of M. Two important conclusions would be drawn from this sort of transformation. That is: 1) it would be possible to translate all results of nonnegative matrices to equivalent Metzler matrices, and; 2) it would follow that λ_o is real and $x_o > 0$ such that $M\,x_o = \lambda_o\,x_o$, and for any other $\lambda \neq \lambda_o$, the Re (λ) < Re (λ_o). This would guarantee a positive and stable equilibrium point for (3) without, of course, any guarantee that it would be easily observable.

At first sight, the equilibrium point of a pure exchange market economy is a unique and stable fixed-point. This finding is also responsible for a real 'déformation professionnelle' in economic thinking. For even when production with delays and time-to-build and increasing returns are added, even when endogenous money and financialization are included, or when faced with nonlinearity and myriads of interconnections (including false signaling), economists remain fixated on an inexorable unique stable fixed-point despite the warning of the Sonnenschein-Mantel-Debreu-Theorem (Sonnenschein, 1972, 1973; Mantel, 1974; Debreu, 1970, 1974). In fact, it is obvious from (3) that the equilibrium $x^* = f\,(x;\, \alpha,\, \omega)$ depends on the distribution of the sets α and ω, i. e., on revealed preferences and supply. Even if M is invertible, changes in budget distributions /or in the supply rate would cause x* to wobble and to elude measurements in the state space since it would be undistinguishable from a transient point. But M is not invertible in the Walrasian system. If we were to write down the augmented matrix M and then use the Gauss-Jordan elimination to find the reduced row-echelon form of the augmented matrix of M, we would find a free variable. Positing x as the price, and knowing that the rank of M is (n-1), then the values of the (n-1) prices would depend on the value assigned to the free variable. That free variable would then be the numéraire, whose arbitrary values would yield infinite stable solutions. This is not all however. All the elements of M are functions of α and ω. Hence, as the distributions of the sets of preferences coefficients and endowments are constantly changing, even with a complete set of data at a given point in time, by the time it would take a super computer to compute x*(.), it would have already changed.

Although non observable, the stable equilibrium assumption gave substance to the Quantity Theory of Money, to Adam Smith's metaphor of the invisible hand as well as to the obsession with a stable equilibrium found in all other results popularized by Bachelier and the Chicago school. As regards the latter point, it should be noted that the deterministic wobbling motion is confused with Brownian motion; that is the first grave error. As we will show shortly, in real market economies, stable equilibria are not guaranteed; that is the second error. And both cast a serious doubt on the validity of the claims transposed *in extenso* to real markets. All that can be said is that the Walrasian construct is a reflexive and therefore controllable system; indeed, the sets of admissible controls are the distributions of α and ω. But in a perfect market each agent has only an infinitesimal influence on the control set, while collections of them are unlikely to act in unison. Hence despite the mathematical reasonableness of the stable equilibrium assumption in that incomplete model, it still cannot even be verified empirically, in particular if n is a large number. What is then the justification to carry it over to a nonlinear and complex system such as a real market economy?

2.2. A Linear Time Invariant Model

The feedback optimization procedure considered here rests on three basic concepts. That is, *multi-inputs-multi-outputs linear time invariant finite-order systems; internally stable feedback; and system norm*. In addition, there is the concept of 'well-posedness' of the optimization problem ensuring that the optimization algorithm does not break down. The aim of the optimization process is to find an LTI feedback controller that stabilizes the feedback system and minimizes the closed-loop system from the exogenous input stream to the cost of producing the output.

Consider now a market economy, E, represented by an LTI model defined by finite dimensional state space model:

$$\dot{x}(t) = A_1\,x\,(t) + A_2\,\omega\,(t) + A_3\,c\,(t) \quad (4)$$

$$o\,(t) = C_1\,x\,(t) + C_2\,\omega\,(t) + C_3\,c\,(t) \quad (5)$$

$$y\,(t) = D_1\,x\,(t) + D_2\,\omega\,(t). \quad (6)$$

Equations (4) to (6) describe an input/output economy E with an input partitioned into 2 vector components, ω (t) and c (t); ω (t) represents a set of exogenous inputs, and c (t) is both another input to E and the output of the controller K. The output of E is also partitioned into 2 vector components, o (t) representing the quadratic financial and thermodynamic costs of producing the output, while y (t) is the output to be measured and to be manipulated, which is also an input to the controller K. System E has the properties of linearity and time invariance. Linearity means that if $\omega_1 (t) \rightarrow y_1 (t)$ and $\omega_2 (t) \rightarrow y_2 (t)$, then $a_1 \omega_1 (t) + a_2 \omega_2 (t) \rightarrow a_1 y_1 (t) + a_2 y_2 (t)$ ($a_i \varepsilon \Re_+$). Time invariance, on the other hand, means that if $\omega (t) \rightarrow y (t)$, then $\omega (t - \tau) \rightarrow y (t - \tau)$.

The controller K is an LTI model defined by a finite dimensional state space model of the form:

$$\dot{x}_{k\,(.)} = A_k\, x_k\, (.) + B_k\, y\, (.) \tag{7}$$

$$c_k\, (.) = C_k\, x_k\, (.) + D_k\, y\, (.). \tag{8}$$

The coefficient matrices A_i, C_i and D_i in ((4)- (6)) are expected to be known, while coefficient matrices A_k, B_k, C_k and D_k are to be designed or found by the optimization algorithm.

Equation (6), the input to K, does not include the controller's output c (.). Then system (4) - (6) defines a closed-loop state space model of the form:

$$\dot{x}_{cl}\, (.) = Acl\, x_{cl}\, (.) + B_{cl}\, \omega\, (,) \tag{9}$$

$$o_{cl}\, (.) = C_{cl}\, x_{cl}\, (.) + D_{cl}\, \omega\, (.), \tag{10}$$

where

$$x_{cl} = [x_k\ \ x]^T,\ A_{cl} = \begin{pmatrix} A_1 + A_3\, D_k\, D_1 & A_3 C_k \\ B_k\, D_1 & A_k \end{pmatrix},\ B_{cl} = \begin{pmatrix} A_1 + A_3\ D_k\, D_1 & A_3 C_k \\ B_k\, D_1 & A_k \end{pmatrix},\ \begin{matrix} C_{cl} = [C_1 + C_3\, D_2\ \ D_1\ \ C_3\, C_k] \\ D_{cl} = [C_2 + C_3\, D_k\ \ D_2]. \end{matrix}$$

For the controller to be stabilizing, the matrix A_{cl} must be a Hurwitz matrix.

The real-valued functions of the feedback design, specified in (9) and (10) are to be minimized with respect to the controller K, subject to the constraints of well-posedness and stabilization. The H-infinity norm is the task of minimizing the H_∞ norm $\| G \|_\infty$ of the transfer matrix G. The matrices A_i, C_i, D_i must be known but they must also be subject to the following conditions to ensure that they are suitable for the feedback optimization. That is, i) the pair (A_1, A_3) must be *stabilizable*, meaning that there exists a P matrix such that $[A_1 + A_3 P]$ is a Hurwitz matrix; ii) the pair (A_1, D_1] must be *detectable*, meaning that there exists a Q matrix such that $[A_1 + Q\, D_1]$ is a Hurwitz matrix; and iii) the optimization procedure must be minimizing and satisfying the condition of existence of a minimizer (not discussed).

It should be noted at this point that the input set cannot be measured accurately due its sheer size and the presence of intangible inputs for which there is no metrics, and similarly for the output set y (t). Part of the difficulty stems from the fact that $x \varepsilon \Re^n$, $c \varepsilon \Re^m$, and $\omega \varepsilon \Re^q$, while the matrices A_i, C_i, D_i, P, and Q, etc., are unknown. If initial conditions and the rank of these matrices are unknown, we cannot claim to know everything about the dynamic behavior of x (t) from information from output measurement. Put more simply, stabilizability and detectability being the *sine qua non* conditions for the claim of a stable equilibrium, hence, the data requirement is too demanding to demonstrate the existence of a stable equilibrium in a real LTI market system. Furthermore, a market economy with positive and negative feedbacks is almost surely nonlinear.

3. Part II

3.1. Feedback Non-Linear H-Infinity Optimization Theory

This section examines two solution concepts in the theory of robust and optimal control of nonlinear systems based on the Hamilton-Jacobi Equations. These equations are a special case of the Hamilton-Jacobi Bellman equations representing a necessary condition describing extremal geometry in generalizing problems of the calculus of variations. The Hamilton-Jacobi inequality (HJI) plays an important role in the study of various qualitative properties of controlled dynamical systems such as stability, invariance and optimality. If a solution to a certain generalized HJI exists, then it is a sufficient condition for stability. The Hamilton-Jacobi-Isaacs equations (HJIE), on the other hand, are the nonlinear version of the Riccati equation studied in the H_∞-control problem for linear systems. We will focus on the contributions of Aliyu (2011) who summarizes all relevant topics on the subject. In particular, he shows (via state feedback H_∞-control problems for affine nonlinear systems that use the theory of dissipative systems (developed mainly by (van der Schaft, 1991; Bazar and Bernhard, 1995) that significant progress had been made. For, van der Schaft had already shown that for time-invariant affine nonlinear systems that are smooth, the state feedback H_∞-control problem is solvable by smooth feedback if there exists a smooth positive semi-definite solution to a dissipation inequality. The non-affine and affine cases considered by Aliyu will suffice for the present purpose, which is to show the necessary and formidable 'data requirement' faced by the would-be controller of the economy.

3.2. Generalities

During the 1960s and 1970s, economists were encouraged by the World Bank to build large general equilibrium models, which produced mainly insignificant results. During the 1990s onward, economists switched from linear H-infinity control developed by Zames (1981, 546-559; Francis (1987), among others, to the theory of nonlinear H-infinity control based on the efficient solution of the Hamilton-Jacobi equations (HJE) and on Hamilton-Jacobi- Bellman equations (HJBE) that extended the contributions of both Euler and Lagrange. The nonlinear case is mainly the contributions of Isidori (1997; Isidori and Astolfi (1992, pp.1283-1293) and others (Doyle, Glover, *et al*., 1989, pp.831-847; van der Schaft, 1992). In this paper, we will be guided mainly by the work of Aliyu (2011, p.133) who argues that the theory of H-infinity control becomes really useful when faced with a Hamiltonian that is independent of time. In that case, it is then possible to separate the variables in the HJE. Subsequently, it was recognized from the calculus of variation that the variational approach to problems of mechanics could equally be applied to problems of optimal control.

The H-infinity optimization problem is formulated in terms of efficient design of a stabilizing controller K (s) that minimizes the H_∞-norm of the closed-loop transfer matrix ($G_{o\,\omega}$) from the input set ω (t) to the output set o (t) for a given system E, defined by some state-space equations.

The term H_∞-control refers to the mathematical space over which the optimization takes place, which is the space of matrix-valued functions that are analytic and bounded in the open right half of the complex plane. The H_∞-norm, on the other hand, is the maximum singular value of the function over that space. The H_∞ algorithms solve suboptimal controller design problems formulated as that of finding a controller for a given $\rho > 0$ that is capable of achieving the closed-loop L_2-gain $\|G_{o\omega}\| < \rho$ if it exists.

As regards the nonlinear equivalent of the linear H_∞-control problem, van der Schaft has shown that for time-invariant affine nonlinear systems that are smooth, state feedback H_∞-control problems are solvable by smooth feedbacks if there exists a smooth positive semi-definite solution to a dissipative inequality, or equivalently, an infinite horizon HJB-inequality, which is the same as the Hamilton-Jacobi-Isaacs (HJI)-inequality found by Basar. The solution of the output-feedback problem with dynamic measurement feedback for affine nonlinear systems was achieved by Isidori (1997; Ball, Walker, *et al*.,1993, pp.546-559), among others. Most of these developments are succinctly summarized in Aliyu who has also examined in dept a series of nonlinear affine and non-affine H_∞-control problems. We will consider two of Aliyu's problems here. The first, the state feedback problem, represents the kind of problems studied by economists in the 1980s. The second arises when the states of the system are not available for feedback or when the output is used for feedback. It is then called: *Robust output measurement feedback nonlinear H_∞-control*. It is a more elaborated model in the affine category that includes uncertainty and parameter variations. It seems to be a better representation of the real market economic. We now consider the first.

3.3. The Non-affine Case

Consider a system or a market economy E with two types of inputs: ω(t) as a collection of exogenous disturbance inputs, and input c(t) (the output of the controller), which becomes the input to the actuator driving E. The main difference between ω (t) and c (t) is that the controller can manipulate c (t) but not ω (t). E has two outputs: o (t) (the cost performance output), and y (t) (the measured output); the latter is both an output of E and an input of the controller; and both outputs are to be measured and regulated.

The problem here is to find a controller K (s) for the generalized E (s) such that the infinity-norm of the transfer function relating input ω (t) to the performance output o (t) is minimized. The minimum gain is ρ*. If the norm for an arbitrary stabilizing controller is $\rho > \rho^*$, then the E(s) is L_2-gain bounded. In control theory, a system ∑ with input ω (t) and output o (t) is said to have L_2-gain less or equal to ρ if $\forall x\ \varepsilon\ N \subseteq X$, $\exists$ k (x) ($0 < k(x) < \infty$, $k(0) = 0$) such that $\int_o^\infty \|o(t)\| \leq \rho^2 \int_o^\infty \|\omega\|^2 + k(x)$, $\forall t > 0$, $\forall \omega(t)$: $x(0) = 0$, and k (x) is a remaining part of the integrals from t to ∞. This leads to the concepts of available storage and storage function. Then ∑ has L_2-gain $\leq \rho$ if N = X. Applied to economy E, L_2-gain is a performance measure. To solve the H_∞-control problem one starts with a value of ρ and reduce it until ρ*is reached.

To construct a typical state-feedback H_∞-control problem for a general class of non-affine non-linear systems, we follow Aliyu (2011,p. 131). Here, the plant problem is compared to economy E (t) with inputs ω (t) and c (t), and outputs are o (t) and y (t); and the controller K (s) represents a set of policies and technologies. Thus the nonlinear system is defined on some manifold $X \subseteq \Re^n$ containing the origin, expressed in local co-ordinates x_i, i ε n. The state-space equations are:

$$E\,(.) : \begin{cases} \dot{x} = F\,(x;\, \omega,\, c) \\ y = x \\ o = O\,(x,\, c), \quad x\,(t_o) = x_o, \end{cases} \tag{11}$$

where the variation of market price dp /dt is represented by $\dot{x}$, and $x\,(.) \subseteq X$ is the state vector. In addition, $c\,(.)\ \varepsilon\ C \subseteq \Re^q$ is a q-dimensional control input belonging to the set of admissible controls C; $\omega\,(.)\ \varepsilon\ W \subset \Re^s$ is the set of inputs to be tracked, which belong to the set of admissible disturbances; $y\,(.)\ \varepsilon\ \Re^n$ is the measured output of E; and $o\ \varepsilon\ \Re^r$ is the performance output to be controlled. Further, $F\,(x;\, \omega,\, c)$: X x W x C $\to$ Z* is the state dynamics function; $O\,(x,\, c)$: X x C $\to \Re^r$ is the controlled output function, and the controller to be synthesized is referred to as K (.). Finally, the functions F (.), and O (.) are assumed to be smooth C^k $(k \geq 1)$ functions of their arguments, and the point x = 0 is assumed by economists of the 1980s to be the unique equilibrium point for E such that F (0,0,0) = 0, O (0,0) = 0 (see Scheinkman, 1976, pp.11-30; Boldrin and Montruccio, 1986, pp.26-39; Benhabib and Nichimura,1979, pp.421-444; Blatt, 1983).

On the assumption that O (x, c) is linearizable, the matrix $\partial O / \partial c$ has full rank l. Letting T* be the cotangent bundle of dim 2n, the Hamiltonian function for the economy E is: H: T* X x W x C $\to \Re$ as :

$$H\,(x,\, l,\, \omega,\, c) = l^T\, F\,(x;\, \omega,\, c) + (1/2)\|O\,(x,\, c)\|^2 - (1/2)\, \rho^2\|\omega\|^2. \tag{12}$$

Equation (12) is locally concave with respect to ω and locally convex with respect to c near the origin, which is also the equilibrium point. Hence, there exists a unique saddle-point (ω, c) for each (x, l) near the origin zero. From the rank l and the Implicit Function Theorem, there exist smooth functions ω*(x, l) and c*(x, l), defined in the neighborhood of the point (0, 0) such that ω*(0, 0) = 0, c*(0, 0), satisfying:

$$\partial H\,(x,\, l,\, \omega^*(.),\, c\,(.)) / \partial \omega = \partial\,(x,\, l,\, \omega^*(.),\, c^*(.)) / \partial c) = 0. \tag{13}$$

Further, suppose there exists a non-negative C^1 function Z*: X $\to \Re$, satisfying the inequality:

$$H^*(x,\, Z_x^T(.)) = H\,[(x,\, Z_x^T(x),\, \omega^*(x,\, Z_x^T(x),\, c^*(x,\, Z_x^T(x)] \leq 0. \tag{14}$$

Then the feedback law is $\omega^* = \omega\,(x,\, Z_x^T(.))$, $c^* = c\,(x,\, Z_x^T(.))$. Substituting $c^* = c\,(x,\, Z_x^T(.))$ in (11) yields the closed-loop system, satisfying:

$$Z_x^T(x)\, F\,(x,\, \omega,\, c^*(x,\, Z_x^T(x)) + (1/2)\|O\,(x,\, c^*(x,\, Z_x^T(x)\|^2 - (1/2)\, \rho^2\|\omega\|^2 \leq 0, \tag{15}$$

which is dissipative with respect to the supply rate $S\,(\omega,\, o) = (1/2)\,[\rho^2\|\omega\|^2 - \|o\|^2$ with storage function Z in the neighborhood of (x, ω) = (0, 0), and $\rho\ \varepsilon\ \Re_{++}$. In this case and the following one, dissipation with respect to the supply rate means that a part of input energy is dissipated in the form of heat and waste.

Obviously, in a physical system, control engineers would measure the variable (usually a unique signal) with a reasonable accuracy. In a social science, on the other hand, this task is much more difficult. For all intents and purposes, the set ω is infinite and contains intangible elements such as agents' confidence for which there is no metric. As the Hamiltonian is dissipative in conformity with the Second Law of thermodynamics, the function Z: X $\to \Re$ exists, but it and all other functions, including the optimal feedbacks ω*(.), and c*(.), are unknown. Hence, the controls cannot be synthesized to guarantee the existence of a stable equilibrium. Further, the above problem neglects important features of a real market economy. For example, what Aliyu calls 'un-modeled uncertainties' contain parameter variations (already discussed in (3)), and uncertainties arising out of the measurements of certain intangibles such as 'herd behavior', consumers' confidence, etc, that are sets in (-1, 1). Perhaps for all these reasons, the economists that ventured into optimal control never succeeded in either observing or demonstrating empirically the existence of an equilibrium point. To add more realism to (11), we consider another Aliyu's model (p.153), which is also discussed in Frieling *et al*., 1996, pp. 264-269; Anderson *et al*., 1998, pp. 1559-1593).

3.4. The Affine Case

For the more realistic affine case, consider an affine robust measurement feedback nonlinear H_∞- control economy shown in Figure 1. This time, there are 3 inputs to E: The exogenous inputs ω (t), the output of the controller c (t), and the output of the set of uncertainties d (t) that bypasses the controller. Economy E has 3 outputs: o (t); y (t) which is an input to the controller; and b, which is an input to the set of uncertainties. The state-space equations are:

$$E: \begin{cases} \dot{x} = f(x) + \Delta f(x, u, t) + G_1(x)\,\omega + [G_2(x) + \Delta G_2(x, u, t)]\, c \\ o = G_3(x) + G_4(x)\, c \\ y = [G_5(x) + \Delta G_5(x, u, t)] + G_6(x)\,\omega; \;\; x(t_o) = x_o \end{cases} \qquad (16)$$

As before, the state vector is x ε X; c ε C $\subseteq \Re^q$, i. e. a q-dimensional controlled input belonging to the set of admissible controls; ω ε W $\subset \Re^s$ ε L_2 (.); y ε Y $\subset \Re^p$ is the measured output of E; and o ε $\Re^v$ is the cost performance output of E to be controlled. Further, F (x, ω, c): X x W x C → Z* is the state dynamics function; O (x, c): X x C → $\Re^v$ is the controlled output function. The set of parameters that are susceptible to variations over time is u ε U $\subset \Re^s$, while Δf, ΔG_2, ΔG_5 ε Ψ are unknown functions belonging to the set of admissible uncertainties. The real C^∞ functions are:

$$\begin{cases} G_1(x): X \to M_{n\,x\,s}(X); & G_2(x): X \to M_{n\,x\,q}(X) \\ G_3(x): X \to \Re^v; & G_4(x): X \to M_{v\,x\,q}(X) \\ G_5(x): X \to \Re^p; & G_6(x): X \to M_{p\,x\,s}(X). \end{cases} \qquad (17)$$

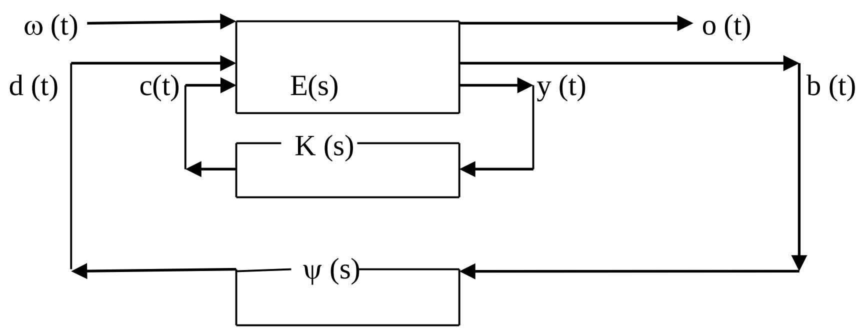

Figure 1. *Robust Measurement Feedback Nonlinear H-infinity Control Economy E.*

These are subject to the following conditions of the system matrices:

$$\text{i) } G_3^T(.)\, G_4(.) = 0 = G_6(.)\, G_3^T(.) \qquad (18)$$
$$\text{ii) } G_4^T(.)\, G_4(.) = I = G_6(.)\, G_6^T(.),$$

where *T* indicates the transpose operation, and I is the identity matrix. Condition i) supposes no feedback between ω(t) and o (t); condition ii) implies that the control weighting matrix is identity for the norm function o (t). It should also be specified that Δf: X → Z*(x), where Z* is the vector space of all C^∞ vector fields in X; ΔG_2 (.) → $M_{n\,x\,q}$ (.), and ΔG_5: X → $\Re^p$.

The task now is to find a dynamic controller for E such that the closed-loop system has L_2-gain (energy) locally from the disturbance input ω (t) to output o (t) that is less or equal to some prescribed ρ* > 0 with internal stability for all admissible (Δf, ΔG_2, ΔG_5) ε Ψ and for all potential parameter variations u ε U ε $\Re^s$. Aliyu has shown that to characterize Ψ some 6 additional matrices of appropriate dimensions are required. For the present purpose it suffices to say that it would be exceedingly difficult, if not impossible, to characterize Ψ in economics.

To solve the affine-robust-measurement-feedback-nonlinear-H_∞-control system, many other conditions must be satisfied, such as observability and zero-state detectability, i. e. both f, and G_3 must be locally detectable. By zero-state observable, it is meant $\exists\ \Omega \subset X$ containing $x_o = 0$ or that any trajectory starting at x_o in Ω, c (t) = 0, y (t) = 0, $\forall t \geq t_o$ implying x (t) = 0. The nonlinear system E is locally zero-state detectable if $\exists$ $N \subset X$ near x = 0 such that $\forall$ x (t_o) ε N if o (t) = 0, c (t) = 0, $\forall t \geq t_o$, implying $\lim_{t \to \infty}$ x (t, t_o, x_o, c) = 0. The system is zero-state detectable if N = X. As it can be seen, there is no hope that these conditions could ever be satisfied for economy E, and there is no point discussing them further, except to say that, more importantly, there must be a smooth positive semi-definite function Z* near the origin that satisfies the Hamilton-Jacobi-Isaacs equation:

$$Z_x^*(.)\, f\,(x) + (1/2)\, Z_x^*(.)\, [(1/\rho^2)\, (G_1(.)\, G_1^T(.) + H_2(x)\, H_2^T(x) - G_2(x)\, G_2^T(x)]\, Z_x^{*T}(.) + (1/2)\, G_3^T(.)\, G_3(.) + (1/2)\, E_1(.)\, E_1^T(.) \leq 0, \tag{19}$$

where H_2(.) and E_1(.) are two of the matrices that characterize the set of admissible uncertainties Ψ.

It should be recalled at this point that our task is not to dwell into the intricacies of stabilizing a controlled economy but to show how difficult it would be to do so. Real market economies do not satisfy the properties of superposition and homogeneity due to friction, adjustment costs, cooperative and competing parts, myriads of interconnections, etc. They are obviously nonlinear and very complex. This is not to say that they are impossible to stabilize, but first optimality would have to be defined and second synthesizing policies in a rivalrous and pluralistic society would have to be found. But it should be borne in mind that, additionally, real modern markets face a measurement problem due to the lack of proper metrics. The data requirement representing myriads of interconnections is visible in the matrices $M_{n\,x\,s}$, $M_{n\,x\,q}$, $M_{v\,x\,q}$, $M_{p\,x\,s}$, and six more needed to characterize the set of uncertainties. All we know is that economy E is a nonlinear dissipative system. It is now well-known that such systems may have multiple stable equilibria, unstable equilibria and a '*strange* 'attractor'; the latter is known to have a countable set of periodic orbits of arbitrarily large period, an uncountable set of aperiodic orbits, and a dense orbit. To assert that economy E tends toward a unique and stable equilibrium on its own power when: a) x_o cannot be assumed to fall in some local stable manifold, or b) the equilibrium cannot be characterized empirically, or c) the system frequently produces undesirable outcomes, reflects "*une grave dėformation professionnelle'*.

4. Concluding Remarks

Orthodox economists are firmly attached to the idea that the economic system, by its very nature, must be a stable system even though no stable market economy has ever been observed. Yet, the notion of stable equilibrium remains the corner stone of both the 'Efficient Market Hypothesis' and the philosophy of neo-liberalism. The collapse of Western economies in 2007-2008 is an additional demonstration of the fallacy of that belief. The question now is that, as a group, economists are well versed in empirical research, why then do they hold such an inalterable belief in unobservable stable equilibria?

This paper attributes this preoccupation to three causes. That is, the Walrasian pure exchange economy; the fact that market economies, being social constructs, are theoretically controllable; and the total neglect of the analyses of complex systems. This paper shows that the Walrasian pure exchange economy, where the notion of stable equilibrium found its first mathematical expression, may be a fine exercise that is nevertheless far-removed from the complexities of areal market economies. Indeed, market economies are social constructs designed to facilitate exchange; they should, therefore, be controllable in theory. The paper then uses the new advances in affine and non-affine nonlinear feedback H-infinity control theory to show that the lack of proper metrics and the data requirements preclude all attempts at empirical verifications. Moreover, market economies are nonlinear systems subject to multiple interconnections, parameter variations, and uncertainties. Their equilibria may be multiple (as ascertained by the Sonnenschein-Mantel-Debreu Theorem), unstable, and deterministically chaotic. All depend on uncertainties and parameter values. Sensitivity to parameter variations, for example, means that minuscule changes here may produce unpredictable and huge undesirable results there. In addition, if the attractors of such systems are non-hyperbolic, then their outputs are extremely sensitive to noise. It then follows that in market economies, where information sets of participants are incomplete, observed outputs contain a noisy component that cannot be filtered out and therefore output measurements are bound to be spurious. Faced with complex systems, it is futile to attempt to establish causes and effects. Rather, it is wiser to start with policies defining the embedding space and then to look for correlates in observed and enduring patterns thrown-off by such systems.

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Willingness to Overpay for Insurance and for Consumer Credit: Search and Risk Behavior Under Price Dispersion

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When income growth under price dispersion reduces the time of search and raises prices of purchases, the increase in purchase price can be presented as the increase in the willingness to pay for insurance or the willingness to pay for consumer credit. The optimal consumer decision represents the trade-off between the propensity to search for beneficial insurance or consumer credit, and marginal savings on insurance policy or consumer credit. Under price dispersion the indirect utility function takes the form of cubic parabola, where the risk aversion behavior ends at the saddle point of the comprehensive insurance or the complete consumer credit. The comparative static analysis of the saddle point of the utility function discovers the ambiguity of the departure from risk-neutrality. This ambiguity can produce the ordinary risk seeking behavior as well as mathematical catastrophes of Veblen-effect's imprudence and over prudence of family altruism. The comeback to risk aversion is also ambiguous and it results either in increasing or in decreasing relative risk aversion. The paper argues that the decreasing relative risk aversion comes to the optimum quantity of money.

Keywords: *consumer search, risk, insurance, credit, optimum quantity of money, Veblen effect, family altruism, mathematical catastrophe.*

JEL Classification: *D11, D81.*

1. Introduction to Indirect Utility Function of Satisficing Optimal Decision

The analysis of the consumption-leisure choice $U=U(Q,H)$ with respect to the wage rate w and to the purchase price reduction and marginal savings got from the search, or to the value $\partial P/\partial S$, can be presented as the static photograph of a step in the dynamic satisficing decision procedure. The ***satisficing*** consumer decision procedure ignores unacceptable high prices P_S; it starts at the reservation level of labor income wL_0 and finishes at the purchase price level $P_P=wL< wL_0$, where the satisficing procedure results in ***optimal decision*** because it equalizes marginal costs of search with its marginal benefit and that equality provides the maximization of the utility function (Malakhov 2014). The use of the truly ***relative price***, i.e., purchase price P_P with regard to the time of search S or to ***the given place of purchase,*** gives new economic explanations for some anomalies of behavior like endowment effect, sunk costs sensitivity, little pre-purchase search of big ticket

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items, and, finally, Veblen effect and money illusion. From the point of view of the problem *maxU(Q,H) subject to* $w/\partial P/\partial S|_{const}=Q/\partial L/\partial S$, where the value $\partial P/\partial S_{const}$ represents the given place of purchase and the value $\partial L/\partial S$ represents the ***propensity to search***, i.e., propensity to substitute labor L for search S, the constraint is created by the core equality of marginal values of search derived from the satisficing decision procedure:

$$w\frac{\partial L}{\partial S}=Q\frac{\partial P}{\partial S} \quad (1)$$

The equilibrium price P_e becomes equal to the sum of consumers' labor costs wL and transaction cost wS, or $P_e=w(L+S)$:

$$\frac{\partial U/\partial H}{\partial U/\partial Q}=-\frac{w}{\partial P/\partial S}\partial^2 L/\partial S\partial H=-\frac{w}{T\partial P/\partial S}=\frac{w}{w(L+S)}=\frac{w}{P_e} \quad (2)$$

where the value $T=1/\partial^2 L/\partial S\partial H$ represent the time horizon until the similar purchase, or the ***commodity lifecycle***.

As we can see, the Equation (2) specifies the paradox formulated by P.Diamond that when search costs are positive the equilibrium price becomes equal to the monopoly price (Diamond 1971). Moreover, the Equation (2) gives another view on home production where G.Becker's model is still the dominant vector of analysis. Indeed, if we consider the household activity to be a specific form of search, the equilibrium price for the final product or the ***willingness to accept*** will be equal to the sum of purchase price of inputs P_P, i.e., of labor costs wL, and transformation costs wS.

Although the original values of the model $\partial P/\partial S$ and $\partial L/\partial S$ look unusual, their modeling tries not to forget the testament of A. Marshall, who told that "*when a great many symbols have to be used, they become very laborious to any one but the writer himself*" (Marshall 1920[1890], p.12). Sometimes such relative values are indispensable, especially when the original G.Stigler's assumption of the diminishing marginal efficiency of search (Stigler 1961) is used ($\partial P/\partial S<0; \partial^2 P/\partial S^2>0$), or when the behavior of the propensity to search is derived (($\partial L/\partial S<0; \partial^2 L/\partial S^2<0$) (Malakhov 2014). However, the understanding of these relative values can be simplified by the graphical illustration of the interrelation between static ($Q_{variable};\partial P/\partial S_{const}$) ***implicit optimal decision*** and dynamic ($Q_{const};\partial P/\partial S_{variable}$) ***explicit satisficing decision*** (Fig.1):

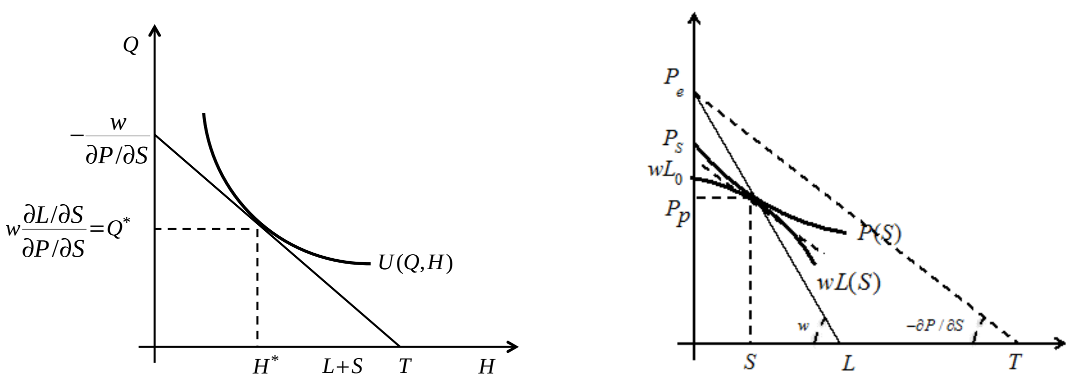

Figure 1. *Implicit optimal decision and explicit satisficing decision*

In addition, the satisficing decision increases real balances because the Equation (1) maximizes the precautionary reserve of money holdings $R(S)=wL(S)-QP(S)$ with respect to the time of search.

The presentation of relatives values in absolute terms, $|\partial P/\partial S|$ and $|\partial L/\partial S|$ simplifies their mathematical treatment without logical losses. This tactic facilitates the comparative static analysis and we can easily derive marginal utilities of money income and money expenditures with respect to optimal values of consumption and leisure (Malakhov 2013):

$$MU_w = \lambda; \qquad (3.1)$$

$$MU_{|\partial P/\partial S|} = -\lambda \frac{w}{|\partial P/\partial S|} \qquad (3.2)$$

The analysis of the second order cross partial derivatives, i.e, the change in the marginal utility of received money income with the change of the place of purchase, or $\partial MU_w / \partial|\partial P/\partial S|$, and the change in the marginal utility (disutility) of the habitual place of purchase with the change in money income, or $\partial MU_{|\partial P/\partial S|} / \partial w$, results in the equation that demonstrates the behavior of the marginal utility of money under the optimal consumption-leisure choice:

$$e_{\lambda,|\partial P/\partial S|} + e_{\lambda,w} = e_{|\partial P/\partial S|.w} - 1 \quad (4)$$

Under the assumption of the diminishing efficiency of search the elasticity of price reduction $e_{|\partial P/\partial S|.w}$ illustrates both the increase in the ***willingness to overpay*** and the decrease in time of search after the increase in the wage rate ($|\partial P_i/\partial S_i|>|\partial P_j/\partial S_j| \Rightarrow P_i>P_j; S_i<S_j$). Hence, it is always positive. When the value of the elasticity of price reduction $e_{|\partial P/\partial S|.w}$ is equal to one, we have

$$e_{\lambda,|\partial P/\partial S|} + e_{\lambda,w} = 0 \quad (5)$$

The Equation (4) also enlightened the way for the comparative static analysis of the indirect utility function where subsequent satisficing decisions optimize consumption-leisure trade-offs with respect to changes in both parts of the constraint. The increase in the wage rate moves consumers from low-price stores to high-price stores. Indeed, the Equation (4) shows us that the indirect utility function depends on two variables in the following manner:

$$v(w,|\partial P/\partial S|) = v(w,|\partial P/\partial S|(w)) \qquad (6)$$

The total derivative of this utility function gives us the following:

$$dv(w,|\partial P/\partial S|(w)) = dw\left(\frac{\partial v}{\partial w}\Big|_{|\partial P/\partial S|const} + \frac{\partial v}{\partial|\partial P/\partial S|}\frac{\partial|\partial P/\partial S|}{\partial w}\right); \quad (7)$$

$$\frac{dv}{dw} = \lambda - \lambda\frac{w}{|\partial P/\partial S|}\frac{\partial|\partial P/\partial S|}{\partial w} = \lambda(1 - e_{|\partial P/\partial S|,w})$$

We see that when the price reduction is unit elastic ($e_{|\partial P/\partial S|.w}=1$), the Equation (5) takes place and the utility stays constant, or $dv/dw=0$. And the following choice of the purchase price which is accompanied by a greater price reduction ($e_{|\partial P/\partial S|.w}>1$) decreases the utility of consumption-leisure choice. The consumption growth is followed by the disproportionally important reduction in leisure time.

2. Willingness to Overpay as Insurance Premium

Usually, guarantees and insurance contracts increase both prices of purchases and price dispersion and we can await that guarantees and insurance contracts raise the equilibrium price reduction $|\partial P/\partial S|$ that equalizes marginal costs of search with its marginal benefit.

We can assume that ***the increase in the wage rate results not in the simple increase in the purchase***

price with respect to the increased income but in the increase in the insurance premium, accompanied by the increase in price reduction. The consumer details his insurance policy and increases the insurance premium with every increase in the wage rate. Our assumption is really ***illustrative*** because here the consumer behaves like a homeowner who raises progressively the fence with any subsequent increase in income. And more insurance policy is detailed, the more efficient is the search, i.e., the greater is the absolute value of the equilibrium price reduction.

The appearance of the saddle point in the utility function gives an answer to the question what the consumer should do in order to avoid the decrease in utility. Obviously, he should decrease relative price reduction i.e., to be… not more modest, but less ambitious with regard to purchase prices after the following increase in the wage rate. We see that the decrease in the willingness to overpay is really possible. The only way to increase both consumption and real balances is not to reduce ***absolute*** overpayments (the value $\partial|\partial P/\partial S|/\partial w$ is always positive) but to reduce ***relative*** overpayments, or to make them less income elastic, i.e., $e_{|\partial P/\partial S|.w}=0{,}9;\ 0{,}8;\ 0{,}7\ldots$ etc., other words, to accept ***incomplete insurance and guarantees*** for items to be bought.

However, this change represents the change in the model of behavior – from risk aversion to risk seeking. Indeed, the prospect theory tells us that facing the inevitable loss, here the decrease in utility, the consumer should take risk (Kahneman and Tversky 1979). Hence, the utility function changes its shape and becomes close to the cubic parabola (Fig.2).

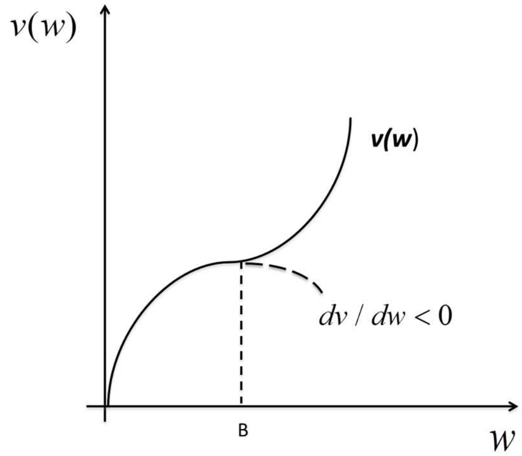

Figure 2. *Utility function under price dispersion*

3. Unwillingness to Overpay for Insurance as Driver of Risk Behavior

When we determine the second derivative of the utility function, we should keep in mind the marginal utility of money income λ as well as the ***unwillingness to overpay*** ($1-e_{|\partial P/\partial S|.w}$) also represent functions of two variables. We can omit labor-intensive intermediate calculations and present the second derivative directly in its total form and in its elasticity form:

$$\frac{d^2v}{dw^2}=\frac{d\lambda}{dw}(1-e_{|\partial P/\partial S|,w})+\lambda\frac{d(1-e_{|\partial P/\partial S|,w})}{dw} \quad (8)$$

$$\frac{d^2v}{dw^2}=\frac{\lambda}{w}(1-e_{|\partial P/\partial S|,w})(e_{\lambda,w}+e_{\lambda,|\partial P/\partial S|}e_{|\partial P/\partial S|,w}+e_{(1-e|\partial P/\partial S|,w),w}) \quad (9)$$

The form of the total second derivative is very useful for the step-by-step analysis of changes in the model of behavior. The elasticity form, although its use is limited by critical points, is helpful in the derivation of the relative measure of risk aversion and in following optional high-order derivations of measures of prudence, which are omitted from the present analysis and left for analysts who are not afraid to work with relative values of the model. Thus, the relative Arrow-Pratt measure takes the following form:

$$\eta=-(e_{\lambda,w}+e_{\lambda,|\partial P/\partial S|}e_{|\partial P/\partial S|,w}+e_{(1-e|\partial P/\partial S|,w),w}) \quad (10)$$

Although we get here the second order elasticity, it is rather simple to understand it. We can denote the value ($1-e_{|\partial P/\partial S|.w}$) as the ***unwillingness to overpay*** and consider its elasticity with respect to the wage rate. When the increase in wage rate decreases the unwillingness to overpay, the second derivative d^2v/dw^2 is strictly negative. Moreover, while the unwillingness to overpay is decreasing ($e_{(1-e|\partial P/\partial S|.w),w}<0$), the absolute value of its elasticity $e_{(1-e|\partial P/\partial S|.w),w}$ is increasing. And with the increase in absolute value of the elasticity of the unwillingness to overpay the relative risk aversion is ***increasing***, i.e., the share of risky assets, i.e., unsecured consumption, is decreasing. Of course, it certainly happens because the subsequent growth in the wage rate and in the equilibrium value of price reduction always results in the increase in real balances, which follow the optimal consumption path of the indirect utility function. It means that the total elasticity of the marginal utility of money is negative, or ($e_{\lambda,w} + e_{\lambda,|\partial P/\partial S|}\, e_{|\partial P/\partial S|.w}) <0$. The last assumption can be verified by the following transformation with the help of the Equation (4):

$$e_{\lambda,w} + e_{\lambda,|\partial P/\partial S|} e_{|\partial P/\partial S|,w} = e_{\lambda,w} + e_{\lambda,|\partial P/\partial S|} e_{|\partial P/\partial S|,w} + e_{\lambda,|\partial P/\partial S|} - e_{\lambda,|\partial P/\partial S|} = (e_{|\partial P/\partial S|,w} - 1)(1 + e_{\lambda,|\partial P/\partial S|}) \quad (11)$$

The price reduction elasticity of the marginal utility of money is positive, or $e_{\lambda,|\partial P/\partial S|}>0$, because it simply states the growth in the marginal utility of money with increase in price of purchase. Hence, the Equation (11) shows us that, when $e_{|\partial P/\partial S|.w} <1$, any increase in wage rate raises real balances and decreases the marginal utility of money because the total elasticity of the marginal utility of money is negative, or ($e_{\lambda,w} +e_{\lambda,|\partial P/\partial S|}\, e_{|\partial P/\partial S|.w}) <0$.

The behavior of the utility function at this stage is described by the following expressions:

$$1-e_{|\partial P/\partial S|.w} >0;\ \lambda >0;\ d\lambda/dw<0;\ de_{(1-e|\partial P/\partial S|.w),w}/dw<0 \Rightarrow d^2v/dw^2<<0 \quad (12)$$

Here the relative risk aversion is increasing because the consumer raises the overpayments or, in the case of insurance, makes the latter more and more detailed. The homeowner begins with insurance for the house and he details it with furniture and paintings. Once there is no object to be insured except the coffer with cash. And the consumer insures it by the following increase in the wage rate and he spends on the coffer's insurance the total increase in income. This action means that neither consumption nor cash kept in the coffer are changed. The insurance policy becomes full or comprehensive. The elasticity of price reduction becomes equal to one ($e_{|\partial P/\partial S|.w}=1$), the unwillingness to overpay becomes equal to zero ($e_{(1-e|\partial P/\partial S|.w),w} =0$), and, according to the Equation (5), the increasing marginal utility of money expenditures completely offsets the decreasing marginal utility of money income:

$$e_{\lambda,w} +e_{\lambda,|\partial P/\partial S|}\, e_{|\partial P/\partial S|.w}= e_{\lambda,|\partial P/\partial S|} + e_{\lambda,w} = 0 \quad (13)$$

This stationary point B also represents the decision node (Fig.2). If the consumer decides to re-insure his comprehensive insurance ($e_{|\partial P/\partial S|.w}>1$) for the given level of consumption, he will decrease his real balances. The utility function will go down ($dv/dw<0$). Thus, the only way to increase both consumption and real balances is to accept incomplete insurance and guarantees for items to be bought.

This decision results in the increase in the unwillingness to overpay $e_{(1-e|\partial P/\partial S|.w),w}$. However, when the increase in the wage rate raises the unwillingness to overpay, the second derivative d^2v/dw^2 becomes positive. The consumer begins to seek risk:

$$1-e_{|\partial P/\partial S|.w} >0;\ \lambda >0;\ d\lambda/dw<0;\ de_{(1-e|\partial P/\partial S|.w),w}/dw>>0; d^2v/ dw^2>0 \quad (14)$$

It happens because at the beginning of risk-seeking the positive ($e_{(1-e|\partial P/\partial S|.w),w}>0$) elasticity of the unwillingness to overpay outweighs the total negative elasticity of the marginal utility of money, or

$$(e_{\lambda,w} +e_{\lambda,|\partial P/\partial S|}\, e_{|\partial P/\partial S|.w}) + e_{(1-e|\partial P/\partial S|.w),w} >0.$$

Here we need some comments on the relationship between real balances and overpayments. The risk-seeking behavior means that the increase in consumption is not well secured. However, the insurance is provided not only by insurance policy but also by real balances, which could represent the ***precautionary savings***. The risk-seeking model of behavior means that the total of precautionary savings and insurance policy is insufficient for the optimal level of consumption. It happens because here the relative increase in real balances is followed by the relative decrease in overpayments. Real balances as the tool of protection of consumption, i.e., of wealth, begin to ***substitute*** overpayments.

Here we come to the question whether precautionary savings and insurance are substitutes or complements. In spite of some analytical solutions of this problem (Ehrlich and Becker (1972)), this question is still open in the general economic analysis. Moreover, when this issue is studied, the attention is usually paid to health and social insurance (Hubbard, Skinner and Zeldes (1995), Guariglia and Rossi (2004)). Here we can only assume the substitutability between money balances and overpayments. The only reason for this assumption is the response of relative overpayments to the continuous decrease in the value of λ, i.e., in the marginal utility of increasing real balances. The economic sense of the decrease in the relative overpayments with respect to the decrease in the marginal utility of money, i.e., in the "price" of money, presumes the substitutability. In addition, the increase in relative overpayments with respect to the decrease in the marginal utility of money presumes that when the consumer is risk-averse, real balances and overpayments becomes complements from the standpoint of the protection of wealth. In any way, the rather harmonic assumption that precautionary savings and insurance are complements in the risk-aversion model and they are substitutes in the risk-seeking model needs, and we are going to see it, more profound analysis.

The comeback from risk seeking to risk aversion is ambiguous. While the positive elasticity of the unwillingness to overpay $e_{(1-e|\partial P/\partial S|.w),w}$ is decreasing, once it certainly matches the total negative elasticity of the marginal utility of money:

$$(e_{\lambda,w} + e_{\lambda,|\partial P/\partial S|}\, e_{|\partial P/\partial S|.w}) + e_{(1-e|\partial P/\partial S|.w),w} = 0 \quad (14)$$

The analysis of the second derivative of the utility function discovers two possible outcomes from the risk neutrality. While the total elasticity of the marginal utility of money is always negative ($e_{\lambda,w} + e_{\lambda,|\partial P/\partial S|}\, e_{|\partial P/\partial S|.w} < 0$), the model of behavior depends here on the decision whether to continue to decrease relative overpayments and to increase the unwillingness to overpay ($e_{(1-e|\partial P/\partial S|.w),w} > 0$), or to increase relative overpayments and to decrease the unwillingness to overpay ($e_{(1-e|\partial P/\partial S|.w),w} < 0$). The continuous increase in real balances with the negative total elasticity of the marginal utility of money ($e_{\lambda,w} + e_{\lambda,|\partial P/\partial S|}\, e_{|\partial P/\partial S|.w} < 0$) provides the negative second derivative $d^2v/dw^2 < 0$ for both outcomes. However, the increase in the unwillingness to overpay, i.e., in the unwillingness to detail insurance policy, results in the "steeper" *sortie* from the risk neutrality. We can verify this fact without laborious calculations of high-order derivatives but with simple back-on-the envelope sketch. The increase in the unwillingness to overpay ($e_{(1-e|\partial P/\partial S|.w),w} > 0$) simply states the fact that the consumer relies more on precautionary savings than on insurance and he increases the share of risky assets, i.e., the share of uninsured commodities or, more precisely, the ***share of commodities with incomplete insurance and guarantees***. Hence, his relative risk aversion becomes decreasing. On the other hand, if he chooses the extension of insurance policy or the decrease in the unwillingness to overpay, he increases his risk aversion. The option to decrease the unwillingness to overpay and to detail insurance policies ($e_{(1-e|\partial P/\partial S|.w),w} < 0$) results in the flat transformation of the utility curve. And with the increasing relative risk aversion the consumer comes again to the next saddle point with the unit elasticity of the price reduction $e_{|\partial P/\partial S|.w} = 1$ that represents the next decision node (Fig.3):

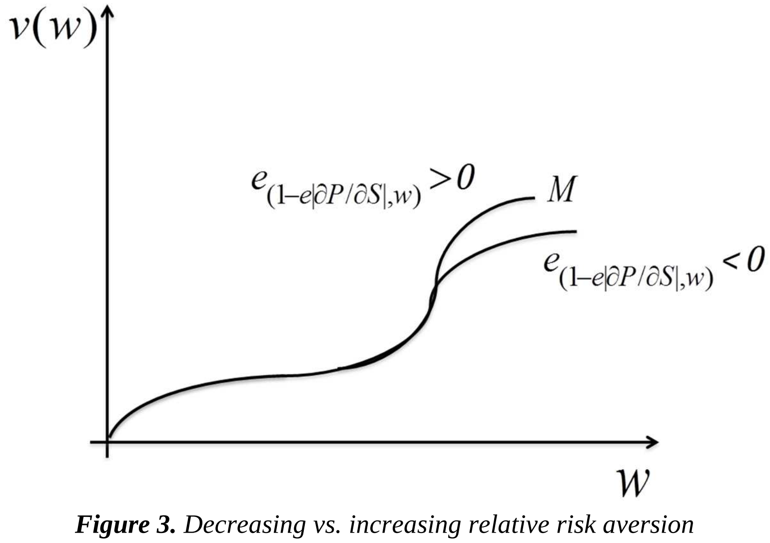

Figure 3. *Decreasing vs. increasing relative risk aversion*

The path of the decreasing relative risk aversion is more intriguing. There, the consumer can continue to decrease relative overpayments until the moment when the value of price reduction $|\partial P/\partial S|$

becomes definitely constant. At this moment the elasticity of the unwillingness to pay $e_{(1-e|\partial P/\partial S|.w),w}$ as well as the elasticity of price reduction $e_{|\partial P/\partial S|.w}$ becomes equal to zero, and the derivatives of the utility function gets its "true" values, or $dv/dw=\lambda$ and $d^2v/dw^2=d\lambda/dw$, i.e., the marginal utility of income becomes unit elastic. Evidently, the marginal utility of money λ is equal here to the opportunity costs of holding cash. However, while the value of price reduction $|\partial P/\partial S|$ doesn't affect here the marginal utility of money in dynamics because its elasticity is equal to zero, it doesn't disappear at all and continues to bother the consumer by its constant value. Here this residual constant $|\partial P/\partial S|$ value can represent the prolongation of the insurance policy for the coffer, leaving all other wealth unsecured.

The insurance for the coffer simply substitutes the costs of illiquidity in the model of the precautionary demand for cash (Whalen 1966, p.316). Thus, the "true" value of money is decreased by the costs of guarding the cash. This assumption corresponds to M.Friedman's reasoning on the optimum quantity of money:

"The amount held will, at the margin, reduce utility – because of concern about the safety of the cash, perhaps, or because of pecuniary costs of storing and guarding the cash." (Friedman 2005 [1969], p.18).

Indeed, if the consumer follows this path once he could come to the point M of the optimum quantity of money. The volume of precautionary saving with respect to consumption becomes so important that it protects the wealth against any disaster. However, if the marginal utility of the optimum quantity of money equals to zero, the consumer doesn't need to insure it.

These considerations raises the question why the consumer cannot change the manner of risk aversion and get the "true" value of money at low levels of income, i.e., why the shift from the increasing to the decreasing risk aversion cannot take place at low values of relative overpayments $e_{|\partial P/\partial S|.w}<<1$. Moreover, it seems that in this case the consumer could avoid saddle points and he could reproduce the exact contour of the Friedman-Savage's utility function (Friedman and Savage 1948). However, in this case high values of the marginal utility of real balances of low-income levels could hardly be offset by the marginal decrease in the unwillingness to overpay and the consumer will come to the saddle point where he will meet "catastrophic" consequences of both imprudence and over prudence.

4. Economic and Mathematical Catastrophes: Veblen Effect and Family Altruism

When G.Becker issued his famous rationalization of family altruism, he stressed the importance of the role of security:

Therefore, altruism helps families insure their members against disasters and other consequences of uncertainty: each member of an altruistic family is partly insured because all other members are induced to bear some of the burden through changes in contributions from the altruist (Becker 1981, pp.3-4).

Hence, the family altruism can be introduced in our model as an additional insurance. There are two possible outcomes for this extra insurance from the saddle point.

We can reproduce the decrease in the individual utility function of the head of the family when relative overpayments really become disproportionate to his individual security, or $e_{|\partial P/\partial S|.w}>1$. The extra insurance is provided by the decrease in real balances ($\partial\lambda/\partial w>0$). However, the following set of equations demonstrates that the decrease in utility ($\partial v/\partial w<0$) is accompanied there not by the risk-seeking behavior but by risk-aversion ($\partial^2 v/\partial w^2<0$). The utility function takes the form of parabola:

$$1-e_{|\partial P/\partial S|.w}>0;\ \lambda>0;\ d\lambda/dw>0;\ de_{(1-e|\partial P/\partial S|.w),w}/dw<<0 \rightarrow d^2v/dw^2<0 \quad (15)$$

Here we could wait for the moment when money balances become equal to zero and the family changes her model of behavior. Unfortunately, in the absence of budget constraints the family could borrow. In this case the marginal utility of money income λ becomes negative. However, when the marginal utility of money income λ becomes negative the head of the family can ***increase*** his utility if he continues to increase overpayments ($\lambda<0;\ (1-e_{|\partial P/\partial S|.w})<0; dv/dw>0$).

Here the head of the family reproduces the Veblen effect. The previous analysis discovered the correspondence between negative marginal utility of money and the extra overpayments (Malakhov 2013)[†]. This is the first "pitfall" the stationary point B prepares for imprudent consumers. Moreover, from the individual point of view the Veblen-effect-like leaving of the saddle point looks more positive than the increase in the unwillingness to overpay. This way can provide more utility until the moment when real balances will be exhausted or the borrowing will be closed and the comeback either to risk aversion or to

[†] If the attribute of the negative marginal utility of money represents a subjective value, the Veblen effect decision could be estimeed from the external satisficing point of view as the decision that decreases the utility.

risk-seeking behavior will take place (Fig.4). In addition, only here we can definitely talk about ***maximizing*** behavior. Indeed, if the aspiration level motivates the consumer to get from the search more than from the labor, i.e., to get marginal savings on purchase greater than the wage rate, the consumer immediately follows the Veblen effect (Malakhov 2013):

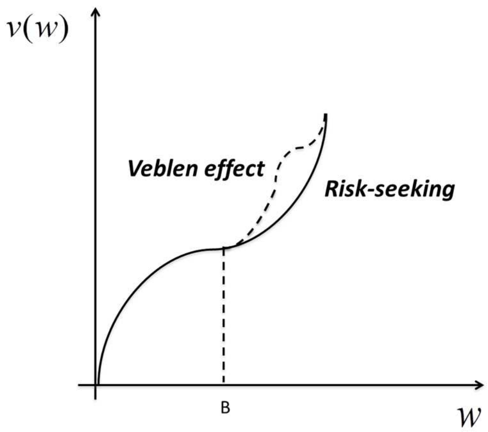

Figure 4. *The option of Veblen effect in risk-seeking behavior*

The equilibrium at the saddle point B is unstable. The consumer can take either maximizing or satisficing decision. The maximizing decision results in the Veblen effect and the satisficing decision produces the ordinary risk seeking behavior. However, the maximizing decision is the decision to purchase a "bad" item with negative marginal utility due to the value $\lambda<0$. The rules of the optimization of consumption-leisure choice stop working, the constraint line takes the north-east direction, and the increase in utility happens only due to an important increase in leisure time that increases the purchase price and compensates the consumption of "bad" item. It really happens when imprudent young family considers holidays on the seaside or in mountains to be vital and parents agree to sponsor vacations for grandchildren. *Hélas*, in the search model of behavior even skiing might become "bad".

The occurrence of Veblen effect with regard to the previous reasoning on the optimum quantity of money tells us that Veblen effect can take place at rather modest levels of income where consumption is far from satiation. However, although this scenario can take place, it does not seem well compatible with the description of the individual utility function within the family. There is another possibility to present family altruism. We can pretend the head of the family to be more "economic man" and to separate altruism from the individual utility function. If we take the factor of ***giving*** as the share of the individual wage rate, we get the following utility function $v^g(w)=v(w)-gw$. However, there we automatically get the other "pitfall" or the mathematical "fold"-type catastrophe due to the existence of the saddle point B and to its unstable equilibrium in the original utility function (Fig.5):

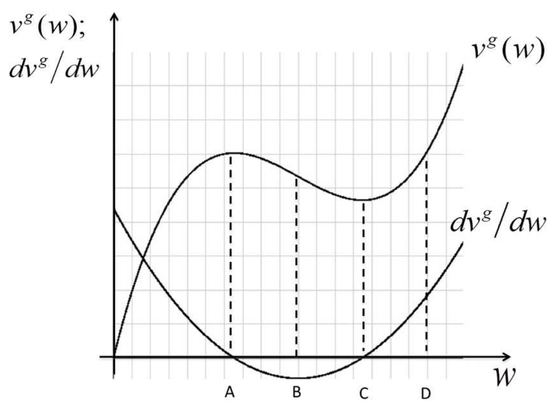

Figure 5. *"Fold" catastrophe of family altruism*

In this case the decrease in the utility function starts at point A when the consumer, the head of the family, is still risk averse and he continues to make protection of his wealth by the increasing real balances and by increasing overpayments. The continuous increase in overpayments discovers the unwillingness of the head of the family to economize. Here, the behavior looks like "pure" altruism. However, once the head of the family changes the model of his behavior and he begins to make risky decisions. It happens at point B when he passes the saddle point of the original utility function with the unit elastic price reduction ($e_{|\partial P/\partial S|.w}=1$). The following increasing unwillingness to overpay gives an idea that the nature of his altruism has been changed. The head of the family becomes more "pragmatic". Although his altruism does not exhausted, his purchase decisions become more prudent. They begin to look like investments. The investments in family reach its peak at point C. Finally, the head of the family begins to feel again the increase in his utility function and at point D he no longer suffers from his altruism, or he finally gets returns on investments:

"Altruistic parents might not have more children than selfish parents, but they invest more in the human capital or quality of children because the utility of altruistic parents is raised by investment returns that accrue to their children." (Becker 1981, p.12).

Indeed, the movement of the utility curve from point A to point D reminds the parental behavior from the birth of a child till the go-out of a young man from the nest. At the beginning parents do not economize on purchases for babies. They are trying to buy everything of high quality and with guarantees. Once, at point B, these purchases take the form of investments, which even in prudent manner lead to point C in the bottom due to their importance. However, the earlier decision at point B to reduce relative overpayments continues to work and finally it pulls out the head of the family from the "pitfall". [‡]

5. Interest rate and willingness to overpay for consumer credit

The common question addressed to the model presented here why it doesn't follow the original G.Stigler's presentation of the equality of marginal values of search with respect to the interest rate. Indeed, the core equation of the model could be presented in that manner:

$$i \times w \frac{\partial L}{\partial S} = Q \frac{\partial P}{\partial S} \quad (16)$$

However, even G.Stigler agreed, that interest rate made "*expected reduction in price…be smaller than the smallest unit of currency*" (Stigler 1961, p.219). While the dynamics of the satisficing decision procedure is short, the model assumes that consumers usually ignore interest rate during the search. If the satisficing consumer doesn't calculate marginal values of search, why he should compute decimals of interest rate and of probabilities?

However, the methodological concern about interest rate can be gratified if we envisage the risk of delay of consumption, i.e., the risk of unexpected rise in prices, and explain overpayments as ***payments for consumer credit***. Other words, interest rate increases price dispersion as well as marginal savings on purchase. The greater is an item under consumer credit, the greater are the marginal savings on this purchase. In this case the comprehensive insurance is transformed into the comprehensive consumer credit and the extra comprehensive insurance ($e_{|\partial P/\partial S|.w}>1$) is transformed into the refinancing of existing debt.

When the consumer buys an item against ***coming*** increase in the wage rate it means that the value $e_{|\partial P/\partial S|.w}=1$ also is coming. This consideration with respect to consumer credit tells us that saddle point with its ***risk neutrality*** is more common economic phenomenon than it was seen from the point of view of insurance. People hold cash for everyday expenses where the cash represents the residual of interest payments. And at $e_{|\partial P/\partial S|.w}=1$ level the total increase in income is going to finance the debt. Neither consumption nor real balances are changed. After that, if the consumer wants to buy another big-ticket item he should either refinance current debt or search this item more intensively in order to decrease relative overpayments, i.e., to find more beneficial credit for the new purchase. The first way decreases the utility and the second way increases risk of unexpected rise in prices during the search.

We remember that while the positive elasticity of the unwillingness to overpay $e_{(1-e|\partial P/\partial S|.w),w}$ is decreasing, once it certainly matches the total negative elasticity of the marginal utility of money and the second derivative of the utility function becomes equal to zero, or $d^2v/dw^2=0$. The following increase in the wage rate again gives a chance to expand consumer credit by the increase in relative overpayments. Facing price uncertainty, the consumer chooses this way of the increasing relative risk aversion. But we already

[‡] When G.Becker cited King Lear's Fool in order to illustrate the Rotten Kid Theorem by the parental willingness to delay contributions until last stage of life he did not take into account the possibility of saddle points in the parental utility function. We have seen that if the consumer continues to increase overpayments without change in the model of behavior at the saddle point his utility goes down infinitely. Once upon a time King Lear simply missed that point. And from the literature point of view it would be better here to remember d'Artagnan-father, who contributed to his son only *quinze écus*, his horse, and some parental advices.

know that real balances at this moment can also protect consumption. If the consumer chooses the decreasing relative risk aversion path, once overpayments become definitely constant. Here, the constant $|\partial P/\partial S|$ value of consumer credit could mean that products are delivered every day by a boy from the neighboring grocery store and once a month the consumer signs a check to the grocer like he renews the insurance policy for the coffer every year.

The constant $|\partial P/\partial S|$ value and the constant place of purchase mean that the consumer is satiated by items that could be bought in other places, i.e., by items that could produce another marginal savings on purchase.

In addition, the consumer also can get the optimum quantity of money but he should decrease for that liquidity costs to the zero level, for example, to give to the grocer a right to debit his current account. With that the consumer reproduces the optimal precautionary model of money holdings – credit is not used, liquidity costs are zero, and the marginal utility of money also equals to zero (Fenestra 1986, p. 283).

However, this theoretical assumption is really illustrative. There are more realistic paths and both of them are well known to us because they represent “catastrophic” solutions. Coming to very low values of the marginal utility of money, either the consumer buys an extraordinary item and, therefore, increases the time of leisure to consume it or he starts the practice of charity that might take a form of the sponsorship for venture investments.

6. Conclusion

The analysis of consumer behavior presented in this paper discovers the methodological power of relative values, which are produced by the process of search. The consumer’s search for beneficial price reduction can be interpreted as the search for reduction in insurance or in interest payments.

The motivation to reduce time of search and to increase quality in consumption after the increase in the wage rate inevitably leads a consumer to the saddle point of the utility function. And the equilibrium in the saddle point is unstable. The consumer can follow maximizing path where he produces the Veblen effect or he can follow common satisficing path where he should take risk. However, even the satisficing path comes to the economic catastrophe of the decrease in utility if consumer takes into account the factor of giving or family altruism.

The model also provides a graphical difference between increasing and decreasing relative risk aversion. The increasing relative risk aversion path could come to the new saddle point of comprehensive insurance or complete consumer credit and the decreasing relative risk aversion could come to the optimum quantity of money.

In addition, this approach can revive the discussion on the optimum quantity of money with an interesting argument. Indeed, when overpayments become constant they could represent not direct interest payments but some fixed expenditures the consumer pays to the government to finance the interest payments on money (Bewley 1983, Mehrling 1995).

The question of the limp-sum taxation leads to the understanding that the model presented here could be useful in the analysis of the optimal taxation. If we substitute in the individual utility function the factor of giving by income tax we also get the “fold”-type catastrophe. However, if one tries to go further and to explain overpayments by VAT or excise tax, the coming trade-off between income taxes and overpayments should be examined with prudence.

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Effects of Measurement Errors on Population Estimates from Samples Generated from a Stratified Population through Systematic Sampling Technique

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In various surveys, presence of measurement errors has led to misleading results in estimation of various population parameters. This study indicates the effects of measurement errors on estimates of population total and population variance when samples are drawn using systematic sampling technique from a stratified population. A finite population was generated through simulation. The population was then stratified into four strata followed by generation of ten samples in each of them using systematic sampling technique. In each stratum a sample was picked at random. The findings of this work indicated that systematic errors affected the accuracy of the estimates by overestimating both the population total and the population variance. Random errors only added variability to the data but their effect on the estimates of the population total and population variance was not that profound.

Keywords: *Measurement errors, Systematic Errors, Stratified population, Population Total and Population Variance*

JEL Classification: *P42, R23*

1. Introduction

The ultimate goal of each survey is to obtain information about the population under study. The theory of sample survey as developed during the past several decades provides us with various kinds of scientific tools for drawing samples and making valid inference about the population parameters of interest. According to Koninj (1973), in measurement of physical quantities the personnel and devices that we have to use may not give as precise measurements as the best available. Measurement errors cannot be completely eliminated but minimized to an extent which their effects on survey results are not that profound. Basic contributions to the methodology of measurement error models were given by Mahalanobis (1946), Hansen (1946) and Sukhatme and Seth (1952) have examined the question of non sampling errors in census and survey work and they have furnished mathematical models for such errors. The objective of this study was to investigate the effects of measurement errors on the estimates of population total and population variance when samples are drawn from a stratified population using systematic sampling technique. The contribution of this study is to establish more weight as to why systematic errors should be minimized if at all valid results are to be obtained.

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2. Systematic sampling

This is a sample selection technique in which sample members are selected from a given population according to a random starting point and a fixed periodic interval. Systematic sampling is still thought of as being random, as long as the periodic interval is determined beforehand and the starting point is random. A common way of selecting members into the sample using systematic sampling is simply by dividing the total number of units in the population by the desired number of units for the sample. The result of the division serves as the marker for selecting sample units from within the given population. Systematic sampling is to be applied only when the given population is logically homogeneous because systematic sample units are uniformly distributed over the population. In some cases systematic sampling is preferred since it spreads the sample more evenly over the population and easier to conduct.

Table 1. *Composition of the k systematic samples.*

Sample no.	1	2	…	i	…	K
	y_1	y_2	$\cdots$	y_i	$\cdots$	y_k
	y_{k+1}	y_{k+2}	$\cdots$	y_{k+i}	$\cdots$	y_{2k}
	…	…	…	…	…	…
	$y_{(n-1)k+1}$	$y_{(n-1)k+2}$	$\cdots$	$y_{(n-1)k+3}$	$\cdots$	y_{nk}
Means	$\bar{y}_1$	$\bar{y}_2$	…	$\bar{y}_i$	…	$\bar{y}_k$

2.1. Stratified Systematic sampling.

This is whereby the finite population under study is divided into relatively homogeneous groups referred to as strata and then systematic sampling is carried out in each stratum to generate samples.

Notations:

N_h - Total number of units in stratum *h* where {h =1, 2, ..., H}

n_h - Number of units in a sample drawn from stratum *h*

μ_{hk} - The true value of the k^{th} unit in stratum *h*.

y_{hk} - is the observed value of the k^{th} in stratum *h*.

Note that $N = N_1 + N_2 + ... + N_H$

3. Sampling design

In the theory of finite population sampling, a sampling design specifies for every possible sample its probability of being drawn. It is convenient to have special notation for this probability which in this case will be P(s).

In other words we assume there is a function P (.) such that P(s) gives the probability of selecting specified samples under the scheme in use. The function P (.) will be referred to as sampling design.

4. The Simple Measurement Model

In this case we would like to formulate a statistical model for measurements made on elements of a sample from a finite population. Consider a finite population, *U = {1…, k,…, N}*.It is assumed that for each element $k \in U$, there exists a true value μ_k and that the objective is to estimate the population total of these true values,

$$t_\mu = \sum_U \mu_k$$

A sample *s* of size n_s is selected from *U* by a given probability sampling design $p(.)$. The idea is to obtain the true value μ_k for each element $k \in s$, but what we actually obtain through the measurement procedure are the observed values y_k for $k \in s$.The observed y_k is composed of the true value μ_k and a random error or both random error and systematic error $y_k - \mu_k$. For lack of better values, y_k is used in the estimation. For a given sample *s*, the random variables $y_k(k \in s)$ are assumed to have a certain joint probability distribution (conditional on s), called a measurement model denoted by *m*. In this case we consider our survey as a two stage process whereby the first stage involves the sample selection, which results in a selected sample *s* and the second stage involves the measurement procedure, which generates an observed value y_k for each $k \in s$. When evaluating expectations and variances with respect to the two stages jointly, the conditional argument is useful. As for the expected values,

$$E_{pm}(.) = E_p\left[E_m(./s)\right]$$

Where $E_m(./s)$ denotes conditional expectation with respect to the measurement model *m*, for a given sample *s*, $E_p(.)$ denotes expectation with respect to the sample design $p(.)$ and $E_{pm}(.)$ denotes expectation with respect to sampling design and measurement model jointly.
Similarly, for the joint variance, called the *pm-variance* or the *total variance*, we have

$$V_{pm}(.) = E_p\left[V_m(./s)\right] + V_p\left[E_m(./s)\right]$$

Where $V_m(./s)$ denotes conditional variance with respect to the model m, given *s*, $V_p(.)$ denotes variance with respect to $P(.)$ and $V_{pm}(.)$ denotes variance with respect to $P(.)$ and *m* jointly.
We specify further the model *m*. For element *k* and *l* belonging to the same sample *s*, the first and second moments are

$$\mu_k = E_m(y_k / s)$$
$$\sigma_k^2 = V_m(y_k / s)$$
$$\text{And} \quad \sigma_{kl} = C_m(y_k, y_l / s)$$

5. Measurement errors.

Measurement is the basis of any scientific study. All measurements are, however, approximate values (not true values) within the limitation of measuring device, measuring environment, process of measurement and human error. Several measurements of the same quality on the same subject will not in general be the same.

Measurement errors refer to errors in survey responses arising from the method of data collection, the respondent or the questionnaire. They include the errors in a survey response as a result of respondents confusion, ignorance, carelessness, or dishonesty; the errors attributable to the interviewer, perhaps as a consequence of the poor or inadequate training, prior expectations regarding respondents' response etc. These measurement errors end up causing a considerable effect on survey estimates. These errors are broadly classified in two categories which are systematic errors and random errors

Systematic errors are biases in measurement which lead to the situation where the mean of many separate measurements differs significantly from the actual value of the measured attribute. All measurements are prone to systematic errors, often of several different types. The errors of this category are characterized by deviation in one direction from the true value. Systematic errors may result from; Usage of faulty instrument, Usage of faulty measuring process and Personal bias. Clearly this type of error cannot be minimized by repeated measurements. Systematic errors can therefore lead to either overestimation or underestimation of the desired population parameters.

Random errors are errors in measurement that lead to measurable values being inconsistent when repeated measures of a constant attribute or quantity are taken. Random errors unlike systematic errors are not unidirectional i.e. some measurements are higher while some are lower than the true value. Another distinguishing aspect of random errors is that it is not biased. It is normally present because of the limitation of the instrument in hand and the limitation of the part of the human ability. No human being can repeat an action in exactly the same manner. Hence it is likely that the same person reports different values with the same instrument, which measures quality correctly. This error is caused by any factor that randomly affects the sample. Random errors add variability to the data but do not affect the average performance for the group. This is why at times it is regarded as 'noise'.

5.1. Mathematical model for measurement errors.

According to Cochran (1977), we assume a large number of independent repetitions of the measurement on the *i*th unit are possible. Let $y_{i\alpha}$ be the value obtained in the α^{th} repetition.

Then $y_{i\alpha} = \mu_i + e_{i\alpha}$

Where μ_i = true value

$e_{i\alpha}$ = measurement error

Where the expectation of e_{ij} is zero and variance is σ_i^2.

$$E\left[e_{i\alpha}\right] = 0$$

$$Var\left[e_{i\alpha}\right] = \sigma_i^2$$

Therefore,

$$E\left[y_{i\alpha} / i\right] = \mu_i$$

$$V\left[y_{i\alpha} / i\right] = \sigma_i^2$$

6. Inclusion probability in a stratified population

Suppose we have a stratified population containing *H* number of strata. If we take one stratum denoted by *h*, our indicator variable becomes,

$$I_{hk}(s) = \begin{cases} 1 \;\; if \;\; the \;\; k^{th} \;\; unit \;\; is \;\; included \;\; in \;\; the \;\; sample \;\; from \;\; stratum \;\; h \\ 0 \;\; otherwise \end{cases}$$

Our first order inclusion probability is denoted by π_{hk} which denotes the probability that element *k* from stratum *h* is included into the sample. In the case of systematic sampling, because each element *k* belongs to one of the $'a_h'$ equally probable systematic samples in stratum '*h*', (where a_h is the sampling interval in stratum *h*.)

$$\pi_{hk} = \frac{1}{a_h}$$

The second order inclusion probability is denoted by π_{hkl} which refers to the probability that both elements '*k*' and '*l*' from stratum *h* are included into the sample. Under systematic sampling in stratum *h*

$$\pi_{hkl} = \begin{cases} \frac{1}{a_h} \;\; if \;\; k \;\; and \;\; l \;\; from \;\; stratum \;\; h \;\; are \;\; in \;\; same \;\; systematic \;\; sample \\ 0 \;\; otherwise \end{cases}$$

The expectation and variance of $I_{hk}(s)$ are obtained as follows,

$$E[I_{hk}(s)] = \pi_{hk}$$
$$V[I_{hk}(s)] = \pi_{hk}(1-\pi_{hk})$$

Covariance

$$\operatorname{cov}[I_{hk}I_{hl}] = \pi_{hkl} - \pi_{hk}\pi_{hl}$$

Since in stratum h our sampling interval is a_h we will have a_h systematic samples in the stratum indicated by s_r where $r = 1, 2, ..., a_h$

7. Horvitz Thompson estimator in a Stratified Population

According to Horvitz and Thompson (1952), the estimator of the population total is given by

$$\hat{t}_{HT} = \sum_{h=1}^{H}\sum_{k=1}^{n_h}\frac{y_{hk}}{\pi_{hk}}$$

Its variance is given by,

$$V\left(\hat{t}_{HT}\right) = \sum_{h=1}^{H}\sum_{k=1}^{N_h}\frac{y_{hk}^2}{\pi_{hk}}(1-\pi_{hk}) + \sum_{h=1}^{H}\sum_{k=1}^{N_h}\sum_{k\neq l}^{N_h}\frac{y_{hk}y_{hl}}{\pi_{hk}\pi_{hl}}(\pi_{hkl} - \pi_{hk}\pi_{hl})$$

According to Sarndal (1992) there is no unbiased estimator of the variance of estimated population total when a sample is generated using systematic sampling technique. A precise estimator below was then chosen

$$\hat{V}\left(\hat{t}_{HT}\right) = \sum_{h=1}^{H}\hat{V}_h\left(\hat{t}_{hHT}\right)$$
$$\hat{V}\left(\hat{t}_{HT}\right) = \sum_{h=1}^{H} N_h^2\,\frac{1-f_h}{n_h}\,S_{hs_r}^2$$
$$where\quad S_{hs_r}^2 = \frac{1}{n_h - 1}\sum_{hs_r}\left(y_{hk} - \bar{y}_{hs_r}\right)$$

Where $f_h = \dfrac{n_h}{N_h}$ represents the sampling fraction

8. The Mean Square Error (MSE)

The mean square error (MSE) of an estimator is one of the many ways to quantify the difference between values implied by an estimator and the true values of the quantity being estimated. The difference occurs because of randomness or because the estimator does not account for information that could produce a more accurate estimate. In order to compare a biased estimate with unbiased estimate or two estimates that have different amounts of bias, a useful criterion is the 'mean square error' of the estimate measured from the population value that is being estimated. Formally,

$$MSE\left(\hat{\mu}\right) = E\left[\hat{\mu} - \mu\right]^2 = E\left[\left(\hat{\mu} - m\right) + (m - \mu)\right]^2$$
$$= E\left(\hat{\mu} - m\right)^2 + 2(m-\mu)E\left(\hat{\mu} - m\right) + (m-\mu)^2$$
$$= \left(Variance\ of\ \ \hat{\mu}\right) + (bias)^2$$

The cross product term varnishes since $E\left(\hat{\mu}-m\right)=0$. The use of the MSE as a criterion of the accuracy of an estimate amounts to regarding two estimates that have the same MSE as equivalent. This is not strictly correct because the frequency distribution of errors $\left(\hat{\mu}-\mu\right)$ of different sizes will not be the same for the two estimates if they have different amounts of bias.

9. Total variance of the estimator of the population total in absence of systematic errors

The total variance in the absence of systematic errors is obtained as follows. Since our population is stratified our measurement model is $y_{hk}=\mu_{hk}+e_{hk}$

Where

$$E_m\left(y_{hk}/s\right)=\mu_{hk} \qquad k\in s$$
$$V_m\left(y_{hk}/s\right)=\sigma_{hk}^2 \qquad k\in s$$
$$Cov_m\left(y_{hk}y_{hl}/s\right)=\sigma_{hkl} \qquad k,l\in s$$

The total variance is given by

$$V_{pm}\left(\hat{t}_{HT}\right)=V_p\left[E_m\left(\hat{t}_{HT}/s\right)\right]+E_p\left[V_m\left(\hat{t}_{HT}/s\right)\right]$$

Where

$$E_p\left[V_m\left(\hat{t}_{HT}\right)\right]=E_p\left[V_m\left(\sum_{h=1}^{H}\sum_{k=1}^{n_h}\frac{y_{hk}}{\pi_{hk}}\right)\right]$$
$$=E_p\left(\sum_{h=1}^{H}\sum_{k=1}^{n_h}\frac{\sigma_{hk}^2}{\pi_{hk}^2}+\sum_{h=1}^{H}\sum_{k,l\in s_{r=1}}^{s_{a_h}}\sum_{k=1}^{n_h}\sum_{k\neq l}^{n_h}\frac{\sigma_{hkl}}{\pi_{hk}\pi_{hl}}\right)$$
$$=\sum_{h=1}^{H}\sum_{k=1}^{N_h}\frac{\sigma_{hk}^2}{\pi_{hk}}+\sum_{h=1}^{H}\sum_{k,l\in s_{r=1}}^{s_{a_h}}\sum_{k=1}^{N_h}\sum_{k\neq l}^{N_h}\frac{\sigma_{hkl}\pi_{hkl}}{\pi_{hk}\pi_{hl}}$$

$$V_p\left[E_m\left(\hat{t}_{HT}\right)\right]=V_p\left[E_m\left(\sum_{h=1}^{H}\sum_{k=1}^{n_h}\frac{y_{hk}}{\pi_{hk}}\right)\right]$$
$$=V_p\left[\sum_{h=1}^{H}\sum_{k=1}^{n_h}\frac{\mu_{hk}}{\pi_{hk}}\right]$$
$$=\sum_{h=1}^{H}\sum_{k=1}^{N_h}\frac{\mu_{hk}^2}{\pi_{hk}}\left(1-\pi_{hk}\right)+\sum_{h=1}^{H}\sum_{k,l\in s_{r=1}}^{s_{a_h}}\sum_{k=1}^{N_h}\sum_{k\neq l}^{N_h}\frac{\left(\pi_{hkl}-\pi_{hk}\pi_{hl}\right)}{\pi_{hk}\pi_{hl}}\mu_{hk}\mu_{hl}$$

Therefore the total variance in absence of systematic errors is given by

$$V_{pm}\left(\hat{t}_{HT}\right)=\sum_{h=1}^{H}\sum_{k=1}^{N_h}\frac{\sigma_{hk}^2}{\pi_{hk}}+\sum_{h=1}^{H}\sum_{k,l\in s_{r=1}}^{s_{a_h}}\sum_{k=1}^{N_h}\sum_{k\neq l}^{N_h}\frac{\sigma_{hkl}\pi_{hkl}}{\pi_{hk}\pi_{hl}}+\sum_{h=1}^{H}\sum_{k=1}^{N_h}\frac{\mu_{hk}^2}{\pi_{hk}}\left(1-\pi_{hk}\right)+\sum_{h=1}^{H}\sum_{k,l\in s_{r=1}}^{s_{a_h}}\sum_{k=1}^{N_h}\sum_{k\neq l}^{N_h}\frac{\left(\pi_{hkl}-\pi_{hk}\pi_{hl}\right)}{\pi_{hk}\pi_{hl}}\mu_{hk}\mu_{hl}$$

By replacing both first and second order inclusion probabilities with $\frac{1}{a_h}$ the variance becomes

$$V_{pm}\left(\hat{t}_{HT}\right)=\sum_{h=1}^{H}\sum_{k=1}^{N_h}a_h\sigma_{hk}^2+\sum_{h=1}^{H}\sum_{k,l\in s_{r=1}}^{s_{a_h}}\sum_{h=1}^{N_h}\sum_{k\neq l}^{N_h}a_h\sigma_{hkl}+\sum_{h=1}^{H}\sum_{k=1}^{N_h}\left(a_h-1\right)\mu_{hk}^2+\sum_{h=1}^{H}\sum_{k,l\in s_{r=1}}^{s_{a_h}}\sum_{k=1}^{N_h}\sum_{k\neq l}^{N_h}\left(a_h-1\right)\mu_{hk}\mu_{hl}$$

10. The mathematical model for measurement of errors in stratified population

Since our population is stratified, the model becomes

$$y_{hk}=\mu_{hk}+e_{hk}$$

μ_{hk} - represents the true value of unit k in stratum h

e_{hk} - represents the measurement error term of unit k in stratum h

$$\therefore E_m\left(y_{hk}/s\right)=b_{hk}+\mu_{hk} \qquad k\in s$$

Where b_{hk} refers to the bias term

$$V_m\left(y_{hk}/s\right)=\sigma_{hk}^2 \qquad k\in s$$

The covariance between k^{th} and l^{th} unit is

$$Cov_m\left(y_{hk}y_{hl}/k,l\in s\right)=E\left[y_{hk}-E\left(Y_{hk}\right)\right]\left[y_{hk}-E\left(y_{hk}\right)\right]$$

$$=E\left(e_{hk}-b_{hk}\right)\left(e_{hl}-b_{hl}\right)$$
$$=Cov\left(e_{hk}e_{hl}\right)$$
$$\therefore Cov_m\left(y_{hk}y_{hl}/k,l\in s\right)=\sigma_{hkl}$$

11. Decomposition of the mean square error.

In this case we will consider the Horvitz Thompson estimator and the effect of measurement errors on its accuracy. We will decompose the mean square error into components, assuming that the measurements obey the simple measurement model ‘m’ as stated before. The mean square error of $\hat{t}_{HT}$ can be written as the sum of the total variance and squared bias.

$$MSE_{pm}\left(\hat{t}_{HT}\right)=V_{pm}\left(\hat{t}_{HT}\right)+\left[B_{pm}\left(\hat{t}_{HT}\right)\right]^2$$

The total variance is given by $V_{pm}\left(\hat{t}_{HT}\right)=E_{pm}\left[\hat{t}_{HT}-E_{pm}\left(\hat{t}_{HT}\right)\right]^2$

The bias is given by $B_{pm}\left(\hat{t}_{HT}\right)=E_{pm}\left(\hat{t}_{HT}\right)-t_{\mu}$

This is called the measurement bias, which arises when expected measurement values on elements do not agree with true values. Variance term can be decomposed as follows

$$V_{pm}\left(\hat{t}_{HT}\right)=V_p\left[E_m\left(\hat{t}_{HT}/s\right)\right]+E_p\left[V_m\left(\hat{t}_{HT}/s\right)\right]=V_1+V_2$$

The first component V_1 is referred to as the sampling variance which is zero in the case of complete enumeration whereas the second component V_2 is referred to as the measurement variance. The Horvitz Thompson estimator for the population total in this case will be

$$\hat{t}_{HT} = \sum_{h=1}^{H}\sum_{k=1}^{n_h}\frac{y_{hk}}{\pi_{hk}}$$

But,

$$E_{pm}\left(\hat{t}_{HT}\right) = E_p\left[E_m\left(\sum_{h=1}^{H}\sum_{k=1}^{n_h}\frac{y_{hk}}{\pi_{hk}}\right)\right]$$

$$= E_p\left(\sum_{h=1}^{H}\sum_{k=1}^{n_h}\frac{g_{hk}}{\pi_{hk}}\right) \quad where \quad g_{hk} = \mu_{hk} + b_{hk}$$

$$= \sum_{h=1}^{H}\sum_{k=1}^{N_h}\frac{g_{hk}}{\pi_{hk}}E_p\left[I_{hk}\right]$$

$$= \sum_{h=1}^{H}\sum_{k=1}^{N_h} g_{hk} \qquad \sin ce \quad E_p\left[I_{hk}\right] = \pi_{hk}$$

$$= \sum_{h=1}^{H}\sum_{k=1}^{N_h}\mu_{hk} + \sum_{h=1}^{H}\sum_{k=1}^{N_h} b_{hk}$$

From the decomposition of the variance term, we have $V_1 \& V_2$ where

$$V_1 = V_p\left[E_m\left(\hat{t}_{HT}\right)\right] \qquad V_2 = E_p\left[V_m\left(\hat{t}_{HT}\right)\right]$$

$$\therefore V_1 = V_p\left[E_m\left(\hat{t}_{HT}\right)\right] = V_p\left[E_m\left(\sum_{h=1}^{H}\sum_{k=1}^{n_h}\frac{y_{hk}}{\pi_{hk}}\right)\right]$$

$$= V_p\left[\sum_{h=1}^{H}\sum_{k=1}^{n_h}\frac{g_{hk}}{\pi_{hk}}\right]$$

$$= \sum_{h=1}^{H}\sum_{k=1}^{N_h}\frac{g_{hk}^2}{\pi_{hk}^2}V\left[I_{hk}\right] + \sum_{h=1}^{H}\sum_{k,l\in s_{r=1}}^{s_{a_h}}\sum_{k=1}^{N_h}\sum_{k\neq l}^{N_h}\frac{g_{hk}}{\pi_{hk}}\frac{g_{hl}}{\pi_{hl}}\operatorname{cov}\left(I_{hk}I_{hl}\right)$$

$$But \quad V\left[I_{hk}\right] = \pi_{hk}\left(1-\pi_{hk}\right) \qquad and \qquad \operatorname{cov}\left(I_{hk}I_{hl}\right) = \left(\pi_{hkl} - \pi_{hk}\pi_{hl}\right)$$

$$\therefore V_1 = \sum_{h=1}^{H}\sum_{k=1}^{N_h}\frac{g_{hk}^2}{\pi_{hk}^2}\pi_{hk}\left(1-\pi_{hk}\right) + \sum_{h=1}^{H}\sum_{k,l\in s_{r=1}}^{s_{a_h}}\sum_{k=1}^{N_h}\sum_{k\neq l}^{N_h}\frac{g_{hk}}{\pi_{hk}}\frac{g_{hl}}{\pi_{hl}}\left(\pi_{hkl} - \pi_{hk}\pi_{hl}\right)$$

$$= \sum_{h=1}^{H}\sum_{k=1}^{N_h}\frac{g_{hk}^2}{\pi_{hk}}\left(1-\pi_{hk}\right) + \sum_{h=1}^{H}\sum_{k,l\in s_{r=1}}^{s_{a_h}}\sum_{k=1}^{N_h}\sum_{k\neq l}^{N_h}\frac{\left(\pi_{hkl} - \pi_{hk}\pi_{hl}\right)}{\pi_{hk}\pi_{hl}}g_{hk}g_{hl}$$

$$but \quad g_{hk} = \mu_{hk} + b_{hk} \qquad and \qquad g_{hl} = \mu_{hl} + b_{hl}$$

$$V_1=\sum_{h=1}^{H}\sum_{k=1}^{N_h}\frac{(1-\pi_{hk})}{\pi_{hk}}(\mu_{hk}+b_{hk})^2+\sum_{h=1}^{H}\sum_{k,l\in s_{r=1}}^{s_{a_h}}\sum_{k=1}^{N_h}\sum_{k\neq l}^{N_h}\frac{(\pi_{hkl}-\pi_{hk}\pi_{hl})}{\pi_{hk}\pi_{hl}}(\mu_{hk}+b_{hk})(\mu_{hl}+b_{hl})$$

$$=\sum_{h=1}^{H}\sum_{k=1}^{N_h}\frac{(1-\pi_{hk})}{\pi_{hk}}(\mu_{hk}^2+2\mu_{hk}b_{hk}+b_{hk}^2)+\sum_{h=1}^{H}\sum_{k,l\in s_{r=1}}^{s_{a_h}}\sum_{k=1}^{N_h}\sum_{k\neq l}^{N_h}\frac{(\pi_{hkl}-\pi_{hk}\pi_{hl})}{\pi_{hk}\pi_{hl}}(\mu_{hk}\mu_{hl}+\mu_{hk}b_{hl}+\mu_{hl}b_{hk}+b_{hk}b_{hl})$$

$$V_1=\sum_{h=1}^{H}\sum_{k=1}^{N_h}\frac{(1-\pi_{hk})}{\pi_{hk}}\mu_{hk}^2+\sum_{h=1}^{H}\sum_{k,l\in s_{r=1}}^{s_{a_h}}\sum_{k=1}^{N_h}\sum_{k\neq l}^{N_h}\frac{(\pi_{hkl}-\pi_{hk}\pi_{hl})}{\pi_{hk}\pi_{hl}}\mu_{hk}\mu_{hl}$$
$$+\sum_{h=1}^{H}\sum_{k=1}^{N_h}\frac{(1-\pi_{hk})}{\pi_{hk}}b_{hk}^2+\sum_{h=1}^{H}\sum_{k,l\in s_{r=1}}^{s_{a_h}}\sum_{k=1}^{N_h}\sum_{k\neq l}^{N_h}\frac{(\pi_{hkl}-\pi_{hk}\pi_{hl})}{\pi_{hk}\pi_{hl}}b_{hk}b_{hl}$$
$$+2\left[\sum_{h=1}^{H}\sum_{k=1}^{N_h}\frac{(1-\pi_{hk})}{\pi_{hk}}\mu_{hk}b_{hl}+\sum_{h=1}^{H}\sum_{k,l\in s_{r=1}}^{s_{a_h}}\sum_{k=1}^{N_h}\sum_{k\neq l}^{N_h}\frac{(\pi_{hkl}-\pi_{hk}\pi_{hl})}{\pi_{hk}\pi_{hl}}\mu_{hk}b_{hl}\right]$$

$$V_2=E_p\left[V_m\left(\hat{t}_{HT}\right)\right]$$
$$=E_p\left[V_m\left(\sum_{h=1}^{H}\sum_{k=1}^{n_h}\frac{y_{hk}}{\pi_{hk}}\right)\right]$$
$$=\sum_{h=1}^{H}\sum_{k=1}^{N_h}\frac{\sigma_{hk}^2}{\pi_{hk}}+\sum_{h=1}^{H}\sum_{k,l\in s_{r=1}}^{s_{a_h}}\sum_{k=1}^{N_h}\sum_{k\neq l}^{N_h}\frac{\sigma_{hkl}\pi_{hkl}}{\pi_{hk}\pi_{hl}}$$

Thus the total variance in presence of measurement errors is

$$V_{pm}\left(\hat{t}_{HT}\right)=V_p\left[E_m\left(\hat{t}_{HT}/s\right)\right]+E_p\left[V_m\left(\hat{t}_{HT}/s\right)\right]=V_1+V_2$$

$$=\sum_{h=1}^{H}\sum_{k=1}^{N_h}\frac{\sigma_{hk}^2}{\pi_{hk}}+\sum_{h=1}^{H}\sum_{k,l\in s_{r=1}}^{s_{a_h}}\sum_{k=1}^{N_h}\sum_{k\neq 1}^{N_h}\frac{\sigma_{hkl}\pi_{hkl}}{\pi_{hk}\pi_{hl}}$$
$$+\sum_{h=1}^{H}\sum_{k=1}^{N_h}\frac{(1-\pi_{hk})}{\pi_{hk}}\mu_{hk}^2+\sum_{h=1}^{H}\sum_{k,l\in s_{r=1}}^{s_{a_h}}\sum_{k=1}^{N_h}\sum_{k\neq l}^{N_h}\frac{\pi_{hkl}-\pi_{hk}\pi_{hl}}{\pi_{hk}\pi_{hl}}\mu_{hk}\mu_{hl}$$

$$+\sum_{h=1}^{H}\sum_{k=1}^{N_h}\frac{(1-\pi_{hk})}{\pi_{hk}}b_{hk}^2+\sum_{h=1}^{H}\sum_{k,l\in s_{r=1}}^{s_{a_h}}\sum_{k=1}^{N_h}\sum_{k\neq l}^{N_h}\frac{\pi_{hkl}-\pi_{hk}\pi_{hl}}{\pi_{hk}\pi_{hl}}b_{hk}b_{hl}$$

$$+2\left(\sum_{h=1}^{H}\sum_{k=1}^{N_h}\frac{(1-\pi_{hk})}{\pi_{hk}}\mu_{hk}b_{hk}+\sum_{h=1}^{H}\sum_{k,l\in s_{r=1}}^{s_{a_h}}\sum_{k=1}^{N_h}\sum_{k\neq l}^{N_h}\frac{\pi_{hkl}-\pi_{hk}\pi_{hl}}{\pi_{hk}\pi_{hl}}\mu_{hk}b_{hl}\right)$$

Under systematic sampling in populations with stratification as previously obtained the inclusion probabilities are:

$$\pi_{hk}=\frac{1}{a_h}$$

$$\pi_{hkl}=\begin{cases}\frac{1}{a_h} & if\ \ k\ \ and\ \ l\ \ from\ \ stratum\ \ h\ \ are\ \ in\ \ same\ \ systematic\ \ sample\\ 0 & if\ \ k\ \ and\ \ l\ \ from\ \ stratum\ \ h\ \ are\ \ in\ \ different\ \ systematic\ \ samples\end{cases}$$

We then substitute these probabilities in

$$V_{pm}\left(\hat{t}_{HT}\right)=V_p\left[E_m\left(\hat{t}_{HT}/s\right)\right]+E_p\left[V_m\left(\hat{t}_{HT}/s\right)\right]$$

$$=\sum_{h=1}^{H}\sum_{k=1}^{N_h}\frac{\sigma_{hk}^2}{\pi_{hk}}+\sum_{h=1}^{H}\sum_{k,l\in s_{r=1}}^{s_{a_h}}\sum_{k=1}^{N_h}\sum_{k\neq 1}^{N_h}\frac{\sigma_{hkl}\pi_{hkl}}{\pi_{hk}\pi_{hl}}$$

$$+\sum_{h=1}^{H}\sum_{k=1}^{N_h}\frac{(1-\pi_{hk})}{\pi_{hk}}\mu_{hk}^2+\sum_{h=1}^{H}\sum_{k,l\in s_{r=1}}^{s_{a_h}}\sum_{k=1}^{N_h}\sum_{k\neq l}^{N_h}\frac{\pi_{hkl}-\pi_{hk}\pi_{hl}}{\pi_{hk}\pi_{hl}}\mu_{hk}\mu_{hl}$$

$$+\sum_{h=1}^{H}\sum_{k=1}^{N_h}\frac{(1-\pi_{hk})}{\pi_{hk}}b_{hk}^2+\sum_{h=1}^{H}\sum_{k,l\in s_{r=1}}^{s_{a_h}}\sum_{k=1}^{N_h}\sum_{k\neq l}^{N_h}\frac{\pi_{hkl}-\pi_{hk}\pi_{hl}}{\pi_{hk}\pi_{hl}}b_{hk}b_{hl}$$

$$+2\left(\sum_{h=1}^{H}\sum_{k=1}^{N_h}\frac{(1-\pi_{hk})}{\pi_{hk}}\mu_{hk}b_{hk}+\sum_{h=1}^{H}\sum_{k,l\in s_{r=1}}^{s_{a_h}}\sum_{k=1}^{N_h}\sum_{k\neq l}^{N_h}\frac{\pi_{hkl}-\pi_{hk}\pi_{hl}}{\pi_{hk}\pi_{hl}}\mu_{hk}b_{hl}\right)$$

We then replace both the first and second order inclusion probabilities with $\frac{1}{a_h}$ to obtain

$$
\begin{aligned}
V_{pm}\left[\hat{t}_{HT}\right] = {} & \sum_{h=1}^{H}\sum_{k=1}^{N_h}\frac{\sigma_{hk}^2}{\left[\frac{1}{a_h}\right]} + \sum_{h=1}^{H}\sum_{k,l\in s_{r=1}}^{s_{a_h}}\sum_{k=1}^{N_h}\sum_{k\neq l}^{N_h}\frac{\sigma_{hkl}\left[\frac{1}{a_h}\right]}{\left[\frac{1}{a_h}\right]\left[\frac{1}{a_h}\right]} \\
& + \sum_{h=1}^{H}\sum_{k=1}^{N_h}\frac{\left[1-\frac{1}{a_h}\right]}{\left[\frac{1}{a_h}\right]}\mu_{hk}^2 + \sum_{h=1}^{H}\sum_{k,l\in s_{r=1}}^{s_{a_h}}\sum_{k=1}^{N_h}\sum_{k\neq l}^{N_h}\frac{\left[\left[\frac{1}{a_h}\right]-\left[\frac{1}{a_h}\right]\left[\frac{1}{a_h}\right]\right]}{\left[\frac{1}{a_h}\right]\left[\frac{1}{a_h}\right]}\mu_{hk}\mu_{hl} \\
& + \sum_{h=1}^{H}\sum_{k=1}^{N_h}\frac{\left[1-\frac{1}{a_h}\right]}{\left[\frac{1}{a_h}\right]}b_{hk}^2 + \sum_{h=1}^{H}\sum_{k,l\in s_{r=1}}^{s_{a_h}}\sum_{k=1}^{N_h}\sum_{k\neq l}^{N_h}\frac{\left[\left[\frac{1}{a_h}\right]-\left[\frac{1}{a_h}\right]\left[\frac{1}{a_h}\right]\right]}{\left[\frac{1}{a_h}\right]\left[\frac{1}{a_h}\right]}b_{hk}b_{hl} \\
& + 2\left[\sum_{h=1}^{H}\sum_{k=1}^{N_h}\frac{\left(1-\frac{1}{a_h}\right)}{\frac{1}{a_h}}\mu_{hk}b_{hk} + \sum_{h=1}^{H}\sum_{k,l\in s_{r=1}}^{s_{a_h}}\sum_{k=1}^{N_h}\sum_{k\neq l}^{N_h}\frac{\left(\frac{1}{a_h}-\frac{1}{a_h}\frac{1}{a_h}\right)}{\frac{1}{a_h}\frac{1}{a_h}}\mu_{hk}b_{hk}\right]
\end{aligned}
$$

This then simplifies to

$$
\begin{aligned}
V_{pm}\left(\hat{t}_{HT}\right) = {} & \sum_{h=1}^{H}\sum_{k=1}^{N_h}a_h\sigma_{hk}^2 + \sum_{h=1}^{H}\sum_{k,l\in s_{r=1}}^{s_{a_h}}\sum_{k=1}^{N_h}\sum_{k\neq l}^{N_h}a_h\sigma_{hkl} \\
& + \sum_{h=1}^{H}\sum_{k=1}^{N_h}\left(a_h-1\right)\mu_{hk}^2 + \sum_{h=1}^{H}\sum_{k,l\in s_{r=1}}^{s_{a_h}}\sum_{k=1}^{N_h}\sum_{k\neq l}^{N_h}\left(a_h-1\right)\mu_{hk}\mu_{hl} \\
& + \sum_{h=1}^{H}\sum_{k=1}^{N_h}\left(a_h-1\right)b_{hk}^2 + \sum_{h=1}^{H}\sum_{k,l\in s_{r=1}}^{s_{a_h}}\sum_{k=1}^{N_h}\sum_{k\neq l}^{N_h}\left(a_h-1\right)b_{hk}b_{hl} \\
& + 2\left[\sum_{h=1}^{H}\sum_{k=1}^{N_h}\left(a_h-1\right)\mu_{hk}b_{hk} + \sum_{h=1}^{H}\sum_{k,l\in s_{r=1}}^{s_{a_h}}\sum_{k=1}^{N_h}\sum_{k\neq l}^{N_h}\left(a_h-1\right)\mu_{hk}b_{hl}\right]
\end{aligned}
$$

12. Simulation of a finite population

Visual basic programming language along with Microsoft access were used to generate a finite population of size N=1000. A population was simulated containing true values ranging from 20 to 90 inclusive. The true values had a normal distribution with a mean of 55 and a variance of 100. The population was then subdivided into four strata. This was done by first arranging all the population units in ascending order. The first 300 units were selected to constitute the first stratum and the following 250 units were selected to constitute the second stratum. The third and fourth strata were also selected to contain 250 and 200 units respectively. A sampling interval of 10 was used in each stratum resulting to 10 systematic samples in each stratum. In each stratum a sample was selected at random. In this case an assumption was made that systematic error b_{hk} is proportional to the true value $b_{hk} = d\mu_{hk}$

13. Results

Table 2. *Estimated population total subjected to systematic errors with proportions of size'd'*

PARAMETER: POPULATION TOTAL AND POPULATION VARIANCE								
		TRUE VALUES	ESTIMATES WITH SYSTEMATIC ERROR					
			d=0.05	d=0.10	d=0.15	d=0.20	d=0.25	d=0.30
STRATUM	1	13320	14215	14881	15547	16213	16879	17545
	2	13350	14030.5	14698	15365.5	16033	16700.5	17368
	3	15050	16034.5	16787	17539.5	18292	19044.5	19797
	4	13840	14369	15061	15723	16445	17137	17829
Estimated pop Total		55560	58649	61427	64205	66983	69761	72539
population Total		55000	57865.3	60615.3	63365.3	66115.3	68865.3	71615.3
Variance of estimator of population Total		1343800	2462003	2620951	2786619	2959005	3138111	3323935
Estimate of the Variance of pop Total		99688.7	188425	199168	210410	222151	234390	247127

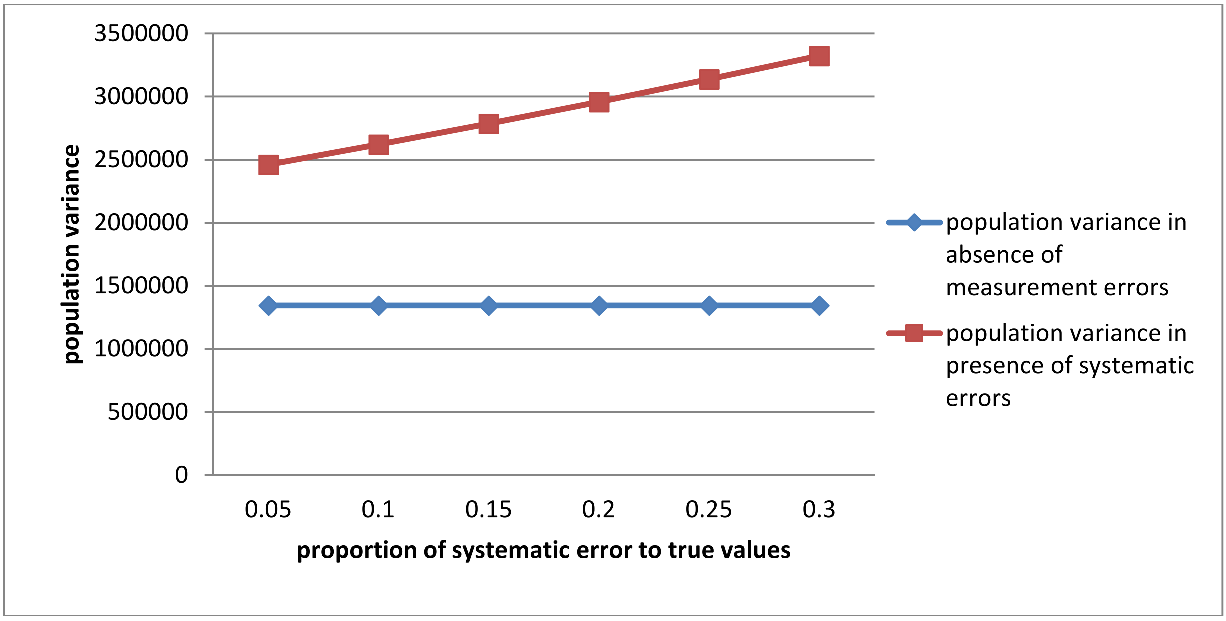

Figure 1. *Effects of systematic errors on variance of estimated population total*

14. Conclusion

The table 2 and figure 1 indicated that there was an increase in population total and population variance with increase in systematic error. The findings of the study indicated that systematic errors had a significant impact on the accuracy of the estimates of both population total and population variance.

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